

The geometric and arithmetic volume of Shimura varieties of orthogonal type

Fritz Hörmann

Mathematisches Institut, Albert-Ludwigs-Universität Freiburg
E-mail: fritz.hoermann@math.uni-freiburg.de

Abstract

We apply the theory of Borcherds products to calculate arithmetic volumes (heights) of Shimura varieties of orthogonal type up to contributions from very bad primes. The approach is analogous to the well-known computation of their geometric volume by induction, using special cycles. A functorial theory of integral models of toroidal compactifications of those varieties and a theory of arithmetic Chern classes of integral automorphic vector bundles with singular metrics are used. We obtain some evidence in the direction of Kudla's conjectures on relations of heights of special cycles on these varieties to special derivatives of Eisenstein series.

Keywords: Kudla's conjectures, Borcherds products, Integral models of toroidal compactifications of mixed Shimura varieties, Arithmetic automorphic vector bundles, Arakelov geometry with singular metrics

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1. Introduction

A Shimura variety of orthogonal type arises from the Shimura datum $\mathbf{O}(L)$ consisting of the orthogonal group $\mathrm{SO}(L_{\mathbb{Q}})$ of a quadratic space $L_{\mathbb{Q}}$ of signature $(m-2, 2)$ and the set

$$\mathbb{D} := \{N \subseteq L_{\mathbb{R}} \mid N \text{ negative definite plane}\},$$

which has the structure of an Hermitian symmetric domain and can be interpreted as a conjugacy class of morphisms $\mathbb{S} = \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \rightarrow \mathrm{SO}(L_{\mathbb{R}})$. For any compact open subgroup $K \subseteq \mathrm{SO}(L_{\mathbb{A}(\infty)})$, we can form the Shimura variety (orbifold):

$$[\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathbb{D} \times \mathrm{SO}(L_{\mathbb{A}(\infty)}) / K].$$

It is a smooth manifold if K is sufficiently small, has a canonical algebraic model over $\mathbb{Z}_{(p)}^1$ for $p \nmid 2D$, where D is the discriminant of a lattice $L_{\mathbb{Z}}$, and K is equal to the corresponding maximal subgroup at p . In the first part of this article (sections 2–6), we recall the functorial theory of

- canonical integral models of toroidal compactifications of mixed Shimura varieties of Abelian type,
- arithmetic automorphic vector bundles on them,

developed in the thesis of the author [18]. This theory, for general Hodge or even Abelian type, still relies on an assumption regarding the stratification of the compactification (6.6) which remains unproven. Even though therefore this

¹if $m > 2$

theory has to be seen as work in progress, we think that it is of independent interest to publish this application.

The orthogonal Shimura varieties are interesting because they easily enable to define algebraic cycles of arbitrary codimension which turn out to have amazing arithmetic properties: For an isometry $x : M \hookrightarrow L$, where M is positive definite, we can simply form

$$\mathbb{D}_x := \{N \in \mathbb{D} \mid N \perp x(M)\}.$$

If $(K\text{-stable})$ lattices $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$ or more generally a K -invariant Schwartz function $\varphi \in S(M_{\mathbb{A}(\infty)}^* \otimes L_{\mathbb{A}(\infty)})$ is chosen, we can form cycles $Z(M, L, \varphi; K)$ on the Shimura variety by taking the quotient of the union of the \mathbb{D}_x over all integral isometries (resp. all isometries in the support of φ in a weighted way) as above. Also for singular lattices M analogous cycles can be defined. Consider a model $M(\frac{K}{\Delta}\mathbf{O}(L))$ of a toroidal compactification of the Shimura variety. We consider its algebraic Chow groups $\mathrm{CH}^n(M(\frac{K}{\Delta}\mathbf{O}(L)))$ and the generating series

$$\Theta_n(L) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^n)^*)} [Z(\mathbb{Z}_Q^n, L, \varphi; K)] \exp(2\pi i Q\tau)$$

with values in the Chow group (tensored with \mathbb{C}). Assume now that $M(\frac{K}{\Delta}\mathbf{O}(L))$ is even defined over \mathbb{Z} in a “reasonably canonical” way. Kudla proposes a “reasonably canonical” way of defining arithmetic cycles $\widehat{Z}(M, L, \varphi; K, \nu)$, depending on the imaginary part ν of τ , too, and also for indefinite M , such that

$$\widehat{\Theta}_n(L) = \sum_{Q \in \mathrm{Sym}^2((\mathbb{Z}^n)^*)} [\widehat{Z}(\mathbb{Z}_Q^n, L, \varphi; K, \nu)] \exp(2\pi i Q\tau),$$

should have values in a suitable Arakelov Chow group $\widehat{\mathrm{CH}}^n(M(\frac{K}{\Delta}\mathbf{O}(L)))_{\mathbb{C}}$. He proposes specific Greens functions, which have singularities at the boundary. The orthogonal Shimura varieties come equipped with a natural Hermitian automorphic line bundle $\Xi^*\overline{\mathcal{E}}$ (whose metric also has singularities along the boundary). Multiplication with a suitable power of its first Chern class and taking push-forward provides us with geometric (resp. arithmetic) degree maps $\deg : \mathrm{CH}^p(\cdots) \rightarrow \mathbb{Z}$ (resp. $\widehat{\deg} : \widehat{\mathrm{CH}}^p(\cdots) \rightarrow \mathbb{R}$). Assuming that an Arakelov theory can be set up to deal with all different occurring singularities, Kudla conjectures (for the geometric case this goes back to Siegel, Hirzebruch, Zagier, Kudla-Millson, Borchers, to name a few)²

- (K1) Θ_n and $\widehat{\Theta}_n$ are (holomorphic, resp. non-holomorphic) Siegel modular forms of weight $\frac{m}{2}$ and genus n .
- (K2) $\deg(\Theta_n)$ and $\widehat{\deg}(\widehat{\Theta}_n)$ are equal to a special value of a normalized version of a standard Eisenstein series of weight $\frac{m}{2}$ associated with the Weil representation of L [17, section 4], resp. its special derivative at the same point.

²If $m - r \leq n + 1$, the statement has to be modified. Here r is the Witt-rank of L .

(K3) $\Theta_{n_1}(\tau_1) \cdot \Theta_{n_2}(\tau_2) = \Theta_{n_1+n_2}(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix})$ and similarly for $\widehat{\Theta}$.

(K4) $\widehat{\Theta}_{m-1}$ can be defined, with coefficients being zero-cycles on the arithmetic model and satisfies the properties above.

Kudla shows (see [23] for an overview) that this implies almost formally vast generalizations of the formula of Gross-Zagier [14]. In full generality the conjectures are known only for Shimura curves [28]. See [17, section 11], the introduction to [18], and especially Kudla's article [23] for more information on what is known in other cases. In the *geometric case* Borcherds [4] shows (K1) for $n = 1$. (K2) follows either from Kudla-Millson theory and the Siegel-Weil formula, or directly from the knowledge of the Tamagawa number, or from Borcherds' theory. All 3 proofs (for non-singular coefficients) are sketched in section 13.

In this work, we show — in a sense — the good reduction part of the *arithmetic case* of (K2) for *all* Shimura varieties of orthogonal type. The approach uses only information from the “Archimedean fibre” to attack the problem. It is a generalization of work of Bruinier, Burgos, and Kühn [8], which dealt with the case of Hilbert modular surfaces. It uses Borcherds' construction of modular forms on these Shimura varieties [3] and a computation of the integral of their norm [7, 22]. More precisely, we show the following:

Main Theorem 14.5. *Assume 6.6.*

Let D' be the product of primes p such that $p^2 \mid D$, where D is the discriminant of $L_{\mathbb{Z}}$. We have

$$\begin{aligned} 1. \quad & \text{vol}_E(M(\frac{K}{\Delta} \mathbf{O}(L_{\mathbb{Z}}))) = 4\widetilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \\ 2. \quad & \widehat{\text{vol}}_{\overline{\mathcal{E}}}(M(\frac{K}{\Delta} \mathbf{O}(L_{\mathbb{Z}}))) \equiv \frac{d}{ds} 4\widetilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)|_{s=0} \quad \text{in } \mathbb{R}_{2D'} \end{aligned}$$

Let $M_{\mathbb{Z}}$ be a lattice of dimension n with positive definite $Q_M \in \text{Sym}^2(M_{\mathbb{Q}}^)$. Let D'' be the product of primes p such that $M_{\mathbb{Z}_p} \not\subseteq M_{\mathbb{Z}_p}^*$ or $M_{\mathbb{Z}_p}^*/M_{\mathbb{Z}_p}$ is not cyclic. Assume*

- $m - n \geq 4$, or
- $m = 4, n = 1$, and $L_{\mathbb{Q}}$ has Witt rank 1.

Then we have

$$\begin{aligned} 3. \quad & \text{vol}_{\mathcal{E}}(Z(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) = 4\widetilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0), \\ 4. \quad & \text{ht}_{\overline{\mathcal{E}}}(Z(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) \equiv \frac{d}{ds} 4 \left(\widetilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \right) \Big|_{s=0} \\ & \text{in } \mathbb{R}_{2DD''}. \end{aligned}$$

3. is true without the restriction on m and n . Note that for $n = 1$ and integral Q_M , we have trivially always $D'' = 1$.

Here \mathbb{R}_N is \mathbb{R} modulo rational multiples of $\log(p)$ for $p \mid N$, and the $\widetilde{\lambda}$ and $\widetilde{\mu}$ are functions in $s \in \mathbb{C}$, given by certain Euler products (11.5) associated with representation densities of $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$. $\pm \widetilde{\mu}$ appears as the “holomorphic part”

of a Fourier coefficient of the standard Eisenstein series associated with the Weil representation of $L_{\mathbb{Z}}$. K is the discriminant kernel and $M(\frac{K}{\Delta}\mathbf{O}(L_{\mathbb{Z}}))$ is any toroidal compactification of the orthogonal Shimura variety (see below). $\overline{\mathcal{E}}$ is the integral tautological bundle on the compact dual equipped with a metric on the restriction of its complex fibre to \mathbb{D} ; the volumes/heights are computed w.r.t. the associated arithmetic automorphic line bundle (10.4) $\Xi^*\overline{\mathcal{E}}$ on $M(\frac{K}{\Delta}\mathbf{O}(L_{\mathbb{Z}}))$. For a more detailed introduction to the method of proof, we refer the reader to section 11 of [17], in which the results of this paper are announced and explained. They have their origin in the thesis of the author [18].

In the first sections we explain in detail the functorial properties integral models of toroidal compactifications of orthogonal Shimura varieties ought to satisfy. These properties have been established in the thesis of the author [18] assuming a certain technical assumption (6.6) that is conjectured to be always satisfied. This is proven if $m \leq 5$ by Lan [30] (the Shimura varieties — Spin-version — are then of P.E.L. type) and was announced in general. For the rest of this work, we assume that the assumption holds true, and we have hence models with the required properties. These are constructed locally (i.e. over an extension of $\mathbb{Z}_{(p)}$). Input data for the theory are p -integral mixed Shimura data (p -MSD) $\mathbf{X} = (P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$ consisting of an affine group scheme $P_{\mathbf{X}}$ over $\text{spec}(\mathbb{Z}_{(p)})$ of a certain rigid type (P) (see 2.3) and a set $\mathbb{D}_{\mathbf{X}}$ which comes equipped with a finite covering $h_{\mathbf{X}} : \mathbb{D}_{\mathbf{X}} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X}, \mathbb{C}})$ onto a conjugacy class in the latter group, subject to some axioms, which are roughly Pink's mixed extension [36] of Deligne's axioms for a pure Shimura datum. Then call a compact open subgroup $K \subseteq P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ admissible, if it is of the form $K^{(p)} \times P_{\mathbf{X}}(\mathbb{Z}_p)$ for a compact open subgroup $K^{(p)} \subseteq P_{\mathbf{X}}(\mathbb{A}^{(\infty, p)})$. For the toroidal compactification a certain rational polyhedral cone decomposition Δ is needed. We call the collection ${}^K\mathbf{X}$ (resp. $\frac{K}{\Delta}\mathbf{X}$) extended (compactified) p -integral mixed Shimura data (p -EMSD, resp. p -ECMSD). These form categories where morphisms $\frac{K}{\Delta}\mathbf{X} \rightarrow \frac{K'}{\Delta'}\mathbf{Y}$ are pairs (α, ρ) of a morphism α of Shimura data and $\rho \in P_{\mathbf{Y}}(\mathbb{A}^{(\infty, p)})$ satisfying compatibility with the K 's and Δ 's. The theory is then a functor M from p -ECMSD to the category of Deligne-Mumford stacks over reflex rings (above $\mathbb{Z}_{(p)}$) which over \mathbb{C} and restricted to p -MSD becomes naturally isomorphic to the one given by the analytic mixed Shimura variety construction. It is characterized uniquely by Deligne's canonical model condition, Milne's extension property (integral canonicity) and a stratification of the boundary into mixed Shimura varieties, together with boundary isomorphisms of the formal completions along the latter with similar completions of more mixed Shimura varieties. These boundary isomorphisms, for the case of the symplectic Shimura varieties, are given by Mumford's construction [12, Appendix]. There is also a functor 'compact' dual from the category of p -MSD to the category of schemes over reflex rings. The duals come equipped with an action of the group scheme $P_{\mathbf{X}}$, and we have morphisms

$$\Xi_{\mathbf{X}} : M(\frac{K}{\Delta}\mathbf{X}) \rightarrow [M^{\vee}(\mathbf{X})/P_{\mathbf{X}, \mathcal{O}_{\mathbf{X}}}] ,$$

which form a pseudo-natural transformation of functors with values in Artin stacks over reflex rings, which are a model of the usual construction over \mathbb{C} if Δ

is trivial. This is the theory of integral automorphic vector bundles. It is compatible with boundary isomorphisms. For more information on the ‘philosophy’ of these objects in terms of motives see the introduction to [18].

In section 7 and 8 a precise general q -expansion principle is derived from these abstract properties.

In section 9 the structure of the models of orthogonal Shimura varieties is investigated and special cycles are defined.

In section 10 we define integral Hermitian automorphic vector bundles, define the notions of arithmetic and geometric volume. Furthermore we set up an Arakelov theory which has enough properties to deal with singularities of the natural Hermitian metrics on automorphic vector bundles. This uses work of Burgos, Kramer, and Kühn [9, 10].

In section 11 we use the general q -expansion principle to prove that Borcherds products with their natural norm yield *integral* sections of an appropriate integral Hermitian line bundle. Among other things the product expansions of Borcherds’ are adelized and their Galois properties investigated.

In section 12, we prove that the bundle of vector valued modular forms for the Weil representation (which appear as input forms in the construction of Borcherds products) has a rational structure. Then we use this to construct input forms with special properties which will be needed in the application.

In section 13 we develop the main ingredient of the proof of the Main Theorem which connects Borcherds’ theory and Arakelov geometry on the orthogonal Shimura varieties.

In section 14 the Main Theorem is proven.

In 3 appendices additional material on “lacunarity of modular forms”, on quadratic forms, and on semi-linear representations is provided.

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2. Group schemes of type (P)

2.1. Our definition of integral mixed Shimura data involves a special kind of group schemes over $\mathrm{spec}(\mathbb{Z}_{(p)})$ which are — in a sense — as rigid as reductive group schemes. We call them group schemes of type (P), and they are a slight generalization of those of type (R) and (RR) considered in [15, XXII, 5.1], which occur as parabolics of reductive group schemes. We use extensively the notation and results of [loc. cit.]. Proofs of the theorems in this section can be found in [18, section 1].

2.2. For a coherent sheaf on a scheme S , we denote by $\mathbb{W}(E)$ its additive group scheme. For a group scheme X over S we denote the functor $D(X) := \underline{\mathrm{Hom}}(X, \mathbb{G}_m)$. For an ordinary Abelian group M , denote M_S the associated constant group scheme.

Definition 2.3. Let P be a group scheme of finite type over S . We call P of type (P) , if the following conditions are satisfied:

There exists a closed unipotent normal subgroup scheme W (called unipotent radical).

S is covered by etale neighborhoods $S' \rightarrow S$ for which there exist

1. a closed reductive subgroup scheme G of $P_{S'}$ such that $P_{S'}$ is isomorphic to the semi-direct product of $W_{S'}$ with G ,
(Then P is smooth and affine over S , and there exist maximal split tori locally in the etale topology [15, XII, 1.7]),
2. a split maximal torus $T \cong D(M_{S'})$ of G , where M is some lattice,
3. a system of roots [15, XIX, 3.2, 3.6] $R = R_G \dot{\cup} R_W \subset M$, such that:

$$\begin{aligned} \mathrm{Lie}(G) &= \mathrm{Lie}(T_{S'}) \oplus \bigoplus_{r \in R_G} \mathrm{Lie}(G)^r \\ \mathrm{Lie}(W_{S'}) &= \bigoplus_{r \in R_W} \mathrm{Lie}(W_{S'})^r, \end{aligned}$$

4. closed embeddings

$$\exp_r : \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \rightarrow W$$

for all $r \in R_W$, inducing via Lie the inclusion and satisfying for all $S'' \rightarrow S'$, $t \in T(S'')$, $X \in \mathrm{Lie}(G_{S''})^r$:

$$\mathrm{int}(t) \exp_r(X) = \exp_r(r(t)X).$$

with the property that any two different $r_1 \in R$, $r_2 \in R_W$ are linearly independent.

Remark 2.4. This property is obviously stable under base change. If the objects required in (1–4) above are already defined over S we call (P, T, M, R) a split group of type (P) .

Group schemes of type (P) have a nice theory of (quasi-)parabolics.

Definition 2.5 ([15, XV, 6.1]). Let P be a group scheme of finite type over S . A **parabolic** subgroup scheme of P is a smooth subgroup scheme Q of G , such that for each $s \in S$, $Q_{\bar{s}}$ is a parabolic subgroup of $G_{\bar{s}}$, (i.e. such that $G_{\bar{s}}/Q_{\bar{s}}$ is proper).

Proposition 2.6. Let S be a reduced scheme, and let (P, T, M, R) be a split group of type (P) over S . If $R' \subset R$ is a closed subset, there exists a unique smooth subgroup scheme $P' \subset P$, containing T , such that

$$\mathrm{Lie}(P') = \mathrm{Lie}(T) \oplus \bigoplus_{r \in R'} \mathrm{Lie}(P)^r$$

P' is of type (P) and closed in P . In particular each parabolic is of this form.

Definition 2.7. Let P be of type (P) . A closed smooth subgroup scheme Q of P is called a **quasi-parabolic** group, if etale locally, say on $S' \rightarrow S$, there is a splitting (P, T, M, R) over S' , $Q_{S'}$ is of the form given in Proposition 2.6 for a $R' \subset R$ which contains a set of positive roots (but not necessarily R_W !).

Theorem 2.8. Let S be reduced and P be a group scheme over S of type (P) .

1. The functor $S' \mapsto \{(\text{quasi-})\text{parabolic subgroups of } P_{S'}\}$ is representable by a smooth (quasi-)projective S -scheme $(Q)\mathcal{PAR}$.
There is an open and closed embedding \mathcal{PAR} into $Q\mathcal{PAR}$.
2. There is an etale sheaf \mathcal{TYPE} of finite sets over S and a surjective morphism

$$\text{type} : Q\mathcal{PAR} \rightarrow \mathcal{TYPE},$$

with the property Q and Q' are locally conjugated in the etale topology, if and only if $\text{type}(Q) = \text{type}(Q')$.

3. If (P, T, M, R) is split

$$\mathcal{TYPE} \cong \left\{ \begin{array}{c} W(R_G)\text{-orbits of closed subsets of } R \\ \text{containing a set of positive roots} \end{array} \right\}_S.$$

4. Let Q be a quasi-parabolic of P . The morphism $P/Q \rightarrow Q\mathcal{PAR}$, $g \mapsto \text{int}(g)Q$ is an open and closed immersion onto a connected component of $Q\mathcal{PAR}$. It is hence the fibre above $\text{type}(Q)$.
5. $\underline{\text{Hom}}(\mathbb{G}_m, P)$ is representable by a smooth affine scheme over S .
There is an etale sheaf \mathcal{FTYPE} of (infinite) sets over S and a surjective morphism

$$\text{ftype} : \underline{\text{Hom}}(\mathbb{G}_m, P) \rightarrow \mathcal{FTYPE},$$

with the property α and α' are locally conjugated in the etale topology, if and only if $\text{ftype}(\alpha) = \text{ftype}(\alpha')$.

6. If (P, T, M, R) is split:

$$\mathcal{FTYPE} \cong \{ W(R_G)\text{-orbits in } M^* \}_S.$$

7. There is a surjective morphism

$$\text{qpar} : \underline{\text{Hom}}(\mathbb{G}_m, P) \rightarrow Q\mathcal{PAR},$$

characterized by the properties

- (a) Let $\alpha : \mathbb{G}_{m, S'} \rightarrow G_{S'}$ be some cocharacter. For each etale $S'' \rightarrow S'$, where there is a splitting $(G_{S''}, T, M, R)$ such that $\alpha_{S''} : \mathbb{G}_{m, S''} \rightarrow T \subset G_{S''}$, we have

$$\text{type}(\text{qpar}(\alpha)) = W(R_G)\{r \in R \mid r \circ \alpha \geq 0\}.$$

- (b) For each $\alpha : \mathbb{G}_{m, S'} \rightarrow G_{S'}$, α factors through $\text{qpar}(\alpha)$.

There is a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{G}_m, P) & \xrightarrow{\mathrm{qpar}} & \mathcal{QPAR} \\ \downarrow \mathrm{ftype} & & \downarrow \mathrm{type} \\ \mathcal{FTYPE} & \longrightarrow & \mathcal{TYPE} \end{array}$$

If (P, T, M, R) is split, the morphism on the bottom is induced by

$$M^* \ni m \mapsto \{r \in R \mid \langle r, m \rangle \geq 0\}.$$

3. Mixed Shimura data

In this section we define p -integral mixed Shimura data. The notion is basically an integral version of the one defined in Pink's thesis [36]. Proofs of the theorems in this section can be found in [18, section 2]. Recall the Deligne torus $\mathbb{S} := \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m)$, with weight morphism $w : \mathbb{G}_{m, \mathbb{R}} \rightarrow \mathbb{S}$.

Definition 3.1. Let P be a group scheme of type (P) defined over $\mathrm{spec}(\mathbb{Z}_{(p)})$, and let $\pi : P \rightarrow P/W$ the projection, where W is its unipotent radical (2.3). A morphism $h : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ is called **admissible** if

- $\pi \circ h$ is defined over \mathbb{R} ,
- $\pi \circ h \circ w$ is a cocharacter of the center of $P_{\mathbf{X}, \mathbb{Q}}/W_{\mathbf{X}, \mathbb{Q}}$ defined over \mathbb{Q} .
- Under the weight filtration on $\mathrm{Lie}(P_{\mathbf{X}, \mathbb{Q}})$ defined by $\mathrm{Ad} \circ h \circ w$, we have:

$$W_{-1}(\mathrm{Lie}(P_{\mathbf{X}, \mathbb{Q}})) = \mathrm{Lie}(W_{\mathbf{X}, \mathbb{Q}}).$$

Recall from [36, Prop. 1.5] that a morphism $h : \mathbb{S} \rightarrow \mathrm{GL}(L_{\mathbb{R}})$ is associated with a mixed Hodge structure if and only if h factors through a subgroup $P_{\mathbf{X}, \mathbb{Q}} \subset \mathrm{GL}(L_{\mathbb{Q}})$, and the induced morphism is admissible.

Lemma 3.2. Let T be a \mathbb{Q} -torus. The following are equivalent conditions:

1. $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}^{(\infty)})$,
2. T is an almost direct product of a \mathbb{Q} -split torus with a torus T' , such that $T'(\mathbb{R})$ is compact.

Definition 3.3. Let p be a prime.

A p -integral mixed Shimura datum (p -MSD) \mathbf{X} consists of

1. a group scheme $P_{\mathbf{X}}$ of type (P) over $\mathrm{spec}(\mathbb{Z}_{(p)})$,
2. a closed unipotent subgroup scheme $U_{\mathbf{X}}$,
3. a homogeneous space $\mathbb{D}_{\mathbf{X}}$ under $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$,
4. a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$ -equivariant finite-to-one morphism $h_{\mathbf{X}} : \mathbb{D}_{\mathbf{X}} \rightarrow \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X}, \mathbb{C}})$, such that the image consists of admissible morphisms (3.1),

subject to the following condition: For (one, hence for all) $h_x, x \in \mathbb{D}_{\mathbf{X}}$,

1. $\text{Ad}_P \circ h_x$ induces on $\text{Lie}(P)$ a mixed Hodge structure of type

$$(-1, 1), (0, 0), (1, -1) \quad (-1, 0), (0, -1) \quad (-1, -1),$$

2. the weight filtration on $\text{Lie}(P_{\mathbb{Q}})$ is given by

$$W_i(\text{Lie}(P_{\mathbb{Q}})) = \begin{cases} \text{Lie}(P_{\mathbb{Q}}) & \text{if } i \geq 0, \\ \text{Lie}(W_{\mathbb{Q}}) & \text{if } i = -1, \\ \text{Lie}(U_{\mathbb{Q}}) & \text{if } i = -2, \\ 0 & \text{if } i < -2, \end{cases}$$

where $W_{\mathbb{Q}}$ is the unipotent radical of $P_{\mathbb{Q}}$ and $U_{\mathbb{Q}} = U_{\mathbf{X}, \mathbb{Q}}$ is a central subgroup,

3. $\text{int}(\pi(h_x(i)))$ induces a Cartan involution on $G_{\mathbb{R}}^{\text{ad}}$, where $G = P/W$,
4. $G_{\mathbb{R}}^{\text{ad}}$ possesses no nontrivial factors of compact type that are defined over \mathbb{Q} ,
5. the center Z of P satisfies the properties of Lemma 3.2.

\mathbf{X} is called **pure**, if $W_{\mathbf{X}} = 1$. A **morphism of p -MSD** $\mathbf{X} \rightarrow \mathbf{Y}$ is a pair of a homomorphism of group schemes $P_{\mathbf{X}} \rightarrow P_{\mathbf{Y}}$ and a homomorphism $\mathbb{D}_{\mathbf{X}} \rightarrow \mathbb{D}_{\mathbf{Y}}$ respecting the maps h .

We call a morphism an **embedding**, if the morphism of group schemes is a closed embedding, and the map $\mathbb{D}_{\mathbf{X}} \rightarrow \mathbb{D}_{\mathbf{Y}}$ is injective.

If we have a p -MSD \mathbf{X} , then an **admissible compact open subgroup** of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ is a group of the form $K^{(p)} P_{\mathbf{X}}(\mathbb{Z}_p)$, where $K^{(p)}$ is a compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty, p)})$.

see [36, 2.1]³ for the rational case. The subgroup scheme $U_{\mathbf{X}}$ is determined by the triple $(P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$.

Definition 3.4. A pair ${}^K \mathbf{X}$ is called **p -integral extended mixed Shimura data** (**p -EMSD**), where \mathbf{X} is p -integral Shimura data, K is an admissible compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$.

A **morphism of p -EMSD** ${}^{K'} \mathbf{Y} \rightarrow {}^K \mathbf{X}$ is a pair (γ, ρ) , where γ is a morphism of p -integral mixed Shimura data and $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty, p)})$ such that $\gamma(K')^{\rho} \subset K$. If γ is an automorphism, then we call the morphism a **Hecke operator**. If γ is an embedding and $\gamma(K_1)^{\rho} = K_2 \cap P_1(\mathbb{A}^{(\infty)})$, and such that the map

$$[P_{\mathbf{Y}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{Y}} \times (P_{\mathbf{Y}}(\mathbb{A}^{(\infty)})/K')] \rightarrow [P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)]$$

is a closed embedding (compare also 3.8), then we call the morphism an **embedding**.

³The property 5. in the definition is stated in [loc. cit.] in a weaker form, namely it is required that the action on W is through a torus of type 3.2. In [31] this property is called (SV5) for pure Shimura varieties.

Lemma 3.5. $P_{\mathbf{X}}(\mathbb{Q}) \cap \text{Stab}(x, P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C}))K$ is finite for every compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ and trivial for sufficiently small K .

Definition 3.6. Let M be an integer. Let $P_{\mathbf{X}}$ be a group scheme over $\mathbb{Z}[1/M]$ of type (P) and $\mathbb{D}_{\mathbf{X}}$ such that they define rational mixed Shimura data \mathbf{X} . For each $p \nmid M$, $P_{\mathbf{X}} \times_{\mathbb{Z}[1/M]} \mathbb{Z}_{(p)}$ will then define p -integral mixed Shimura data which we equally denote by \mathbf{X} . Let $L_{\mathbb{Z}}$ be a lattice with a faithful representation of $P_{\mathbf{X}}$, i.e. a closed embedding $P_{\mathbf{X}} \hookrightarrow \text{GL}(L_{\mathbb{Z}[1/M]})$.

For each integer N , we define the following compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$:

$$K(N) := \{g \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) \mid gL_{\widehat{\mathbb{Z}}} = L_{\widehat{\mathbb{Z}}}, g \equiv \text{id} \pmod{N}\}.$$

If $p \nmid N$ then $K(N)$ is admissible at p .

Lemma 3.7. For each (admissible) compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, there is a $\gamma \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ (resp. $\in P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$) such that

$$P_{\mathbf{X}}^{\gamma} \subseteq K(1).$$

- Lemma 3.8.** 1. For each ${}^{K_1}\mathbf{X}_1$ and an embedding $\alpha : \mathbf{X}_1 \hookrightarrow \mathbf{X}_2$, there is an admissible $K_2 \subset P_{\mathbf{X}_2}(\mathbb{A}^{(\infty)})$ such that ${}^{K_1}\mathbf{X}_1 \hookrightarrow {}^{K_2}\mathbf{X}_2$ is an embedding (3.4).
2. If ${}^{K_1}\mathbf{X}_1 \hookrightarrow {}^{K_2}\mathbf{X}_2$ is an embedding, and if K_2 is neat, then for each $K'_2 \subset K_2$, the map ${}^{K'_2 \cap P_{\mathbf{X}_1}(\mathbb{A}^{(\infty)})}\mathbf{X}_1 \hookrightarrow {}^{K'_2}\mathbf{X}_2$ is an embedding.
3. Let $P_{\mathbf{X}_2}$ operate linearly on a $V_{\mathbb{Z}_{(p)}}$, and choose a lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Z}_{(p)}}$. There is an integer N such that for all $(M, p) = 1, N \mid M$, ${}^{K(M)_1}\mathbf{X}_1 \hookrightarrow {}^{K(M)_2}\mathbf{X}_2$ is an embedding.

We have the following structure theorem for p -integral mixed Shimura data

Proposition 3.9. Let \mathbf{X} be p -MSD.

There are smooth closed subgroup schemes of $P_{\mathbf{X}}$: $W_{\mathbf{X}}$ (the unipotent radical) and $U_{\mathbf{X}} = \mathbb{W}(W_{-2}(\text{Lie}(P_{\mathbf{X}})))$. There is a smooth group scheme $V_{\mathbf{X}} \cong \mathbb{W}(\text{gr}_{-1}(\text{Lie}(P_{\mathbf{X}})))$ and an exact sequence

$$0 \longrightarrow U_{\mathbf{X}} \longrightarrow W_{\mathbf{X}} \longrightarrow V_{\mathbf{X}} \longrightarrow 0.$$

There is a closed reductive subgroup scheme $G_{\mathbf{X}}$, such that $P_{\mathbf{X}} = W_{\mathbf{X}} \rtimes G_{\mathbf{X}}$. Any two such subgroup schemes are conjugated by an element in $W_{\mathbf{X}}(\mathbb{Z}_{(p)})$.

For an morphism of p -MSD $\alpha : \mathbf{X} \rightarrow \mathbf{Y}$, we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & U_{\mathbf{X}} & \longrightarrow & W_{\mathbf{X}} & \longrightarrow & V_{\mathbf{X}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U_{\mathbf{Y}} & \longrightarrow & W_{\mathbf{Y}} & \longrightarrow & V_{\mathbf{Y}} \longrightarrow 0. \end{array}$$

If α is an embedding, the vertical maps are closed embeddings, and the outer ones are given by saturated inclusions (i.e. inducing an inclusion mod p as well) of the corresponding modules over $\mathbb{Z}_{(p)}$. Furthermore, there are closed reductive subgroup schemes $G_{\mathbf{X}}$ and $U_{\mathbf{Y}}$ of $P_{\mathbf{X}}$ and $P_{\mathbf{Y}}$, respectively, such that $P_{\mathbf{X}} = W_{\mathbf{X}} \rtimes G_{\mathbf{X}}$, $P_{\mathbf{Y}} = W_{\mathbf{Y}} \rtimes G_{\mathbf{Y}}$ and such that $G_{\mathbf{X}}$ is mapped to $G_{\mathbf{Y}}$.

Definition/Proposition 3.10. *Let \mathbf{X} be p -MSD. We have the following converse to 3.9:*

Given two $\mathbb{Z}_{(p)}$ -modules V and U acted on by $P_{\mathbf{X}}$ with unimodular invariant symplectic form $\Psi : V \times V \rightarrow U$ (i.e. inducing an isomorphism $V \cong V^$) This defines a group scheme W_0 sitting in an exact sequence:*

$$0 \longrightarrow U_0 \longrightarrow W_0 \longrightarrow V_0 \longrightarrow 0,$$

where $U_0 := \mathbb{W}(U)$ and $V_0 := \mathbb{W}(V)$. Using the action of $P_{\mathbf{X}}$ we may form a semi-direct product $P_{\mathbf{X}'} := W_0 \rtimes P_{\mathbf{X}}$. Assume that every subquotient of $\text{Lie}(W_{0,\mathbb{R}})$ is of type

$$\{(-1, 0), (0, -1)\} \text{ or } \{(-1, -1)\}.$$

Define $\mathbb{D}_{\mathbf{X}'}$ as

$$\{(x, k) \in \mathbb{D}_{\mathbf{X}} \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, P'_{\mathbb{C}}) \mid \\ h_x = \phi \circ k; \pi' \circ k : \mathbb{S}_{\mathbb{C}} \rightarrow (P'/U')_{\mathbb{C}} \text{ is defined over } \mathbb{R}\}.$$

\mathbf{X}' is then p -MSD, called a **unipotent extension** of \mathbf{X} , denoted by $\mathbf{X}[U, V]$.

$\mathbb{D}_{\mathbf{X}'} \rightarrow \mathbb{D}_{\mathbf{X}}$ is a torsor under $W_0(\mathbb{R})(W_0 \cap U_{\mathbf{X}'})(\mathbb{C})$.

We have $\mathbf{X}[U, V]/W_0 \cong \mathbf{X}$.

Proposition 3.9 may be strengthened as follows. Every p -MSD satisfies

$$\mathbf{X} \cong (\mathbf{X}/W_{\mathbf{X}})[\text{Lie}(U_{\mathbf{X}}), \text{Lie}(V_{\mathbf{X}})],$$

the symplectic form and action of $G_{\mathbf{X}}$ being determined by \mathbf{X} .

Definition 3.11. *Let \mathbf{X} be (rational) MSD. For $G_{\mathbf{X},\mathbb{Q}} := P_{\mathbf{X},\mathbb{Q}}/W_{\mathbf{X},\mathbb{Q}}$ every \mathbb{Q} -parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$ is the inverse image of a \mathbb{Q} -parabolic subgroup of $G_{\mathbf{X},\mathbb{Q}}^{\text{ad}}$. Let $G_{\mathbf{X},\mathbb{Q}}^{\text{ad}} = G_{1,\mathbb{Q}} \times \cdots \times G_{r,\mathbb{Q}}$ be the decomposition into \mathbb{Q} -simple factors. Choose \mathbb{Q} -parabolic subgroup $Q_{i,\mathbb{Q}} \subseteq G_{i,\mathbb{Q}}$ for every i , and let $Q_{\mathbb{Q}}$ be the inverse image of $Q_{1,\mathbb{Q}} \times \cdots \times Q_{r,\mathbb{Q}}$ in $P_{\mathbf{X},\mathbb{Q}}$. We call $Q_{\mathbb{Q}}$ an **admissible** \mathbb{Q} -parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$, if every $Q_{i,\mathbb{Q}}$ is either equal to $G_{i,\mathbb{Q}}$ or a maximal proper \mathbb{Q} -parabolic subgroup of $G_{i,\mathbb{Q}}$.*

Lemma 3.12. *If \mathbf{X} is p -MSD and $Q_{\mathbb{Q}}$ is a parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$, then there is a parabolic subgroup scheme Q of $P_{\mathbf{X}}$ (2.5), such that $Q_{\mathbb{Q}} = Q \times_S \text{spec}(\mathbb{Q})$.*

3.13. Let S be a maximal \mathbb{R} -split torus of $(G_{\mathbb{R}})^{\text{der}}$. Let R be the root system of S (acting on $\text{Lie}(G^{\text{ad}})$). The irreducible components of R are of type (C) or (BC) [11, Corollaire 3.1.7], cf. also [1, §2, Proposition 4]. The *long roots* [11, loc. cit.], cf. also [1, p. 185]. $L = \{\alpha_1, \dots, \alpha_g\}$ form a basis of mutually orthogonal roots of S . An admissible \mathbb{R} -parabolic (3.11) $Q_{\mathbb{R}} \supset S$ is determined (in the sense of 2.8) by a homomorphism $\lambda : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow S$, where

$$\langle \lambda, \alpha \rangle = \begin{cases} 2 & \alpha \in L', \\ 0 & \alpha \notin L'. \end{cases}$$

where L' is any subset of L . If the corresponding cocharacter can be defined over \mathbb{Q} , then Q is defined over \mathbb{Q} .

Recall the group $H_0 := \{(z, \alpha) \in \mathbb{S} \times \mathrm{GL}_{2, \mathbb{R}} \mid z\bar{z} = \det(\alpha)\}$. Let $h_0 : \mathbb{S}_{\mathbb{R}} \rightarrow H_{0, \mathbb{R}}$ be the morphism given by $z \mapsto (z, h(z))$, where $h \in \mathbb{D}_{\mathbf{H}_1}$ is any element. Here $\mathbb{D}_{\mathbf{H}_1}$ is the set of representations of type $(-1, 0), (0, -1)$ (see 4.1 for the notation). Let $h_{\infty} : \mathbb{S}_{\mathbb{C}} \rightarrow H_{0, \mathbb{C}}$ be the morphism $z \mapsto (z, h_m(z))$, where $h_m : \mathbb{S}_{\mathbb{C}} \rightarrow \begin{pmatrix} * & * \\ * & * \end{pmatrix}$ describes the mixed Hodge structure having the same Hodge filtration as the one described by h , but weight filtration corresponding to (i.e. fixed by) $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$.

Proposition 3.14. *Let $Q_{\mathbb{Q}}$ be a \mathbb{Q} -parabolic subgroup of $P_{\mathbf{X}, \mathbb{Q}}$. Let π' be the projection $P_{\mathbf{X}} \rightarrow P_{\mathbf{X}}/U_{\mathbf{X}}$. The following are equivalent:*

1. $Q_{\mathbb{Q}}$ is admissible
2. For every $x \in \mathbb{D}_{\mathbf{X}}$ there is a unique homomorphism

$$\omega_x : H_{0, \mathbb{C}} \rightarrow P_{\mathbf{X}, \mathbb{C}}$$

such that

- (a) $\pi' \circ \omega_x : H_{0, \mathbb{C}} \rightarrow (P_{\mathbf{X}}/U_{\mathbf{X}})_{\mathbb{C}}$ is already defined over \mathbb{R} ,
 - (b) $h_x = \omega_x \circ h_0$,
 - (c) $\omega_x \circ h_{\infty} \circ w : \mathbb{G}_{m, \mathbb{C}} \rightarrow Q_{\mathbb{C}}$ is conjugated to $\mu \cdot \lambda$, where λ is the morphism constructed in 3.13, $\mu = h_x \circ w$, and $\mathrm{Lie}(Q_{\mathbb{C}})$ is the direct sum of all nonnegative weight spaces in $\mathrm{Lie}(P_{\mathbf{X}, \mathbb{C}})$ under $\mathrm{Ad}_P \circ \omega_x \circ h_{\infty} \circ w$.
3. There exists an $x \in \mathbb{D}_{\mathbf{X}}$ and a homomorphism ω_x such that the three conditions in 2. are satisfied.

If $P_{\mathbf{X}}$ is reductive, Q corresponds to a boundary component in the sense of [AMRT, III, p. 220, no. 2]. The morphism $\omega_x \circ h_{\infty}$ is independent of the choice of h_0 .

Definition/Theorem 3.15. *Assume that \mathbf{X} is p -MSD. Choose an admissible $Q_{\mathbb{Q}}$ as above. We will define mixed Shimura data \mathbf{B} as follows: Let $P_{\mathbf{B}, \mathbb{Q}}$ be the smallest normal \mathbb{Q} -subgroup of Q , such that $\omega_x \circ h_{\infty}$ factorizes through it. Consider the map*

$$\begin{aligned} \mathbb{D}_{\mathbf{X}} &\rightarrow \pi_0(\mathbb{D}_{\mathbf{X}}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}}) \\ x &\mapsto ([x], \omega_x \circ h_{\infty}). \end{aligned} \tag{1}$$

Choose a $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$ -orbit $\mathbb{D}_{\mathbf{B}}$ containing an $([x], \omega_x \circ h_{\infty})$ in the above image. The image is contained in the union of finitely many such. Each $\mathbb{D}_{\mathbf{B}}$ is a finite covering of the corresponding $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$ -orbit $h(\mathbb{D}_{\mathbf{B}})$ in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}})$. Let $\mathbb{D}_{\mathbf{B}} \rightrightarrows \mathbf{X}$ be its inverse image in $\mathbb{D}_{\mathbf{X}}$.

The closure $P_{\mathbf{B}}$ of $P_{\mathbf{B}, \mathbb{Q}}$ in $P_{\mathbf{X}}$ is of type (P) [18, Theorem 2.4.5], and hence \mathbf{B} is p -integral mixed Shimura data and called a **boundary component** of \mathbf{X} . It is called **proper**, if Q is a proper parabolic, otherwise **improper**.

A **boundary map** $\mathbf{B}' \rightrightarrows \mathbf{X}$ is an isomorphism of \mathbf{B}' with one of the boundary components \mathbf{B} .

For each Q there are finitely many choices of $\mathbb{D}_{\mathbf{B}}$'s and accordingly, finitely many boundary components.

Proposition 3.16. *There is a functorial map, called **projection on the imaginary part**,*

$$\mathrm{im} : \mathbb{D}_{\mathbf{X}} \rightarrow U_{\mathbf{X}}(\mathbb{R})(-1) \quad x \mapsto u_x,$$

where u_x is the unique element, such that $\mathrm{int}(u_x^{-1}) \circ h_x$ is defined over \mathbb{R} .

Proposition 3.17. *If \mathbf{B} is a boundary component of \mathbf{X} and $\alpha : \mathbf{X} \rightarrow \mathbf{X}'$ is a morphism, there is a unique boundary component \mathbf{B}' of \mathbf{X}' and a corresponding map $\tilde{\alpha} : \mathbf{B} \rightarrow \mathbf{B}'$. If α is an embedding, $\tilde{\alpha}$ is either.*

Proposition 3.18. 1. *Let $\mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}}$ be as in Definition 3.15. Let $\mathbb{D}_{\mathbf{X}}^0$ be a connected component of $\mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}}$ and $\mathbb{D}_{\mathbf{B}}^0$ be corresponding component of $\mathbb{D}_{\mathbf{B}}$. Then*

- (a) *The map $\mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}} \rightarrow \mathbb{D}_{\mathbf{B}}$ is an open embedding.*
 - (b) *The image of $\mathbb{D}_{\mathbf{X}}^0$ in $\mathbb{D}_{\mathbf{B}}^0$ is the inverse image of an open complex cone $C := C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \subset U_{\mathbf{B}}(\mathbb{R})(-1)$ under the map $\mathrm{im}|_{\mathbb{D}_{\mathbf{B}}^0}$.*
 - (c) *The cone is an orbit in $U_{\mathbf{B}}(\mathbb{R})(-1)$ under translation by $U_{\mathbf{X}}(\mathbb{R})(-1)$ and conjugation by $Q(\mathbb{R})^\circ$. It is also invariant under translation by $(U_{\mathbf{B}} \cap W_{\mathbf{X}})(\mathbb{R})(-1)$.*
 - (d) *Modulo $(U_{\mathbf{B}} \cap W_{\mathbf{X}})(\mathbb{R})(-1)$ the cone C is a non-degenerate homogeneous self-adjoint cone (in the sense of [AMRT, II, p. 57, §1.1]).*
2. *Consider a morphism of Shimura data $\iota : \mathbf{X} \rightarrow \mathbf{X}'$. For each rational boundary component \mathbf{B} of \mathbf{X} there is a unique boundary component \mathbf{B}' of \mathbf{X}' and a morphism $\tilde{\iota} : \mathbf{B} \rightarrow \mathbf{B}'$ such that*

$$\begin{array}{ccc} \mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}} & \xrightarrow{\iota} & \mathbb{D}_{\mathbf{B}' \Rightarrow \mathbf{X}'} \\ \downarrow & & \downarrow \\ \mathbb{D}_{\mathbf{B}} & \xrightarrow{\tilde{\iota}} & \mathbb{D}_{\mathbf{B}'} \end{array}$$

commutes.

3. *Each boundary component \mathbf{B}' of \mathbf{B} is naturally a boundary component of \mathbf{X} . This defines a partial order on the set of boundary components of \mathbf{X} .*

Definition 3.19. *Let \mathbf{B}_1 be a rational boundary component of \mathbf{X} and $\mathbb{D}_{\mathbf{X}}^0$ be a connected component of $\mathbb{D}_{\mathbf{B}_1 \Rightarrow \mathbf{X}}$. Let $C^*(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_1}) \subset U_{\mathbf{B}_1}(\mathbb{R})(-1)$ denote the union of the cones $C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_2})$ for all rational boundary components \mathbf{B}_2 such that $\mathbf{B}_1 \Rightarrow \mathbf{B}_2 \Rightarrow \mathbf{X}$. It is a convex cone. Form the following quotient*

$$C_{\mathbf{X}} := \coprod_{(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_1})} C^*(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_1}) / \sim$$

by the equivalence relation generated by the graph of all embeddings

$$C^*(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_2}) \hookrightarrow C^*(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_1})$$

*for $\mathbf{B}_1 \Rightarrow \mathbf{B}_2 \Rightarrow \mathbf{X}$. It is called the **conical complex** associated with \mathbf{X} (cf. [36, 4.24]).*

Consider the set

$$C_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)}).$$

$P_{\mathbf{X}}(\mathbb{Q})$ acts on this from the left by conjugation of boundary components [36, 4.23] and on $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ by left multiplication. $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ acts via multiplication on the right on the second factor. Furthermore $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ acts on the second factor through left multiplication.

Let a set Δ of subsets of $C_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ be given, such that every $\sigma \in \Delta$ is contained in some $C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho$. We denote by $\Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho)$ the subset of $\sigma \in \Delta$, such that $\sigma \subset \overline{C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho}$.

Let K be an (admissible) compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$. For any $\mathbb{D}_{\mathbf{X}}^0, \mathbf{B}$ and ρ , we let $\Gamma_U \subset U_{\mathbf{B}}(\mathbb{Q})$ be the image of

$$\{z \in Z(P_{\mathbf{X}}(\mathbb{Q})) \mid z|_{\mathbb{D}_{\mathbf{X}}} = \text{id}\} U_{\mathbf{B}}(\mathbb{Q}) \cap {}^\rho K$$

under the projection $Z(P_{\mathbf{X}}(\mathbb{Q})) \times U_{\mathbf{B}}(\mathbb{Q}) \rightarrow U_{\mathbf{B}}(\mathbb{Q})$.

Definition 3.20. Δ is called a *K-admissible (partial) rational polyhedral cone decomposition* for \mathbf{X} , if

1. For each $\mathbb{D}_{\mathbf{X}}^0, \mathbf{B}$ and ρ , $\Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho)$ is a (partial) rational polyhedral cone decomposition of the closure of $C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho$. We understand a cone as open in its closure.
2. Δ is invariant under right multiplication of K and under left multiplication of $P_{\mathbf{X}}(\mathbb{Q})$.
3. For each \mathbf{B} the set $\bigcup_{\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})} \Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho)$ is invariant under left multiplication of $P_{\mathbf{B}}(\mathbb{A}^{(\infty)})$.

It is called **finite**, if the quotient $P_{\mathbf{X}}(\mathbb{Q}) \backslash \Delta / K$ is finite.

It is called **complete**, if in 1. $\Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho)$ is a complete rational polyhedral cone decomposition.

It is called **projective**, if on each $\Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho)$ there exists a polarization function ([1, IV, §2.1], cf. [12, IV, 2.4]).

It is called **smooth** with respect to K , if for all $\mathbb{D}_{\mathbf{X}}^0, \mathbf{B}$ and ρ , as above, $\Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho)$ is smooth with respect to the lattice Γ_U .

The condition 3. is called the arithmeticity condition. Without it, the compactification exists over \mathbb{C} but may not descend to the reflex field or a reflex ring.

Definition 3.21. A triple ${}^K_{\Delta} \mathbf{X}$ is called *p-integral extended compactified mixed Shimura data (p-ECMSD)*, where everything is as in Definition 3.4, but Δ is in addition a K-admissible (partial) rational polyhedral cone decomposition.

Morphisms of p-ECMSD have to satisfy the property that for each $\sigma_1 \in \Delta_1$ there is a $\sigma_2 \in \Delta_2$ with $\gamma(\sigma_1)^\rho \subset \sigma_2$.

Let ${}^K_{\Delta} \mathbf{X}$ be p-ECMSD and $[\alpha, \rho] : {}^{K'} \mathbf{Y} \rightarrow {}^K \mathbf{X}$ be a morphism of p-EMSD, such that α is a closed embedding.

Set $[\alpha, \rho]^* \Delta$ to be the set of all cones $\{(u, \rho') \mid (\alpha(u), \alpha(\rho')\rho) \in \sigma\}$ for all $\sigma \in \Delta$. This is a K' -admissible rational partial cone decomposition for \mathbf{Y} . It is finite, resp. complete, resp. projective if Δ is finite, resp. complete, resp. projective. This association is functorial. If $[\alpha, \rho]$ was an embedding (this includes a condition on K, K' , see 3.4), we call $\overset{K'}{\Delta} \mathbf{Y} \rightarrow \overset{K}{\Delta} \mathbf{X}$ an **embedding**.

Be aware that, in general, smoothness is not inherited by $[\alpha, \rho]^* \Delta$, but we have:

Proposition 3.22 ([18, Theorem 2.4.12]). *Let \mathbf{X} be p -MSD. Let K be an admissible compact open.*

1. K -admissible (complete) rational polyhedral cone decompositions for \mathbf{X} exist.
2. If Δ is a K -admissible rational polyhedral cone decomposition for \mathbf{X} , there is a smooth and projective refinement Δ' . Any refinement of Δ' will be projective again.
3. If Δ_i , $i = 1, 2$ are 2 rational polyhedral cone decompositions for \mathbf{X} , there is a common refinement Δ (supported on the intersection of their supports).
4. If $\alpha : \mathbf{X} \rightarrow \mathbf{X}'$ is an embedding and $\rho \in P_{\mathbf{X}'}(\mathbb{A}^{(\infty, p)})$ is given, there is K' such that we have an embedding $[\alpha, \rho] : \overset{K}{\Delta} \mathbf{X} \rightarrow \overset{K'}{\Delta'} \mathbf{X}'$, a K' -admissible rational polyhedral smooth and projective cone decomposition Δ' for \mathbf{X}' , with the property that $\Delta := [\alpha, \rho]^* \Delta'$ is smooth and projective. For every smooth refinement of $\tilde{\Delta}$ of Δ , there is a smooth refinement $\tilde{\Delta}'$ of Δ' with $\tilde{\Delta} := [\alpha, \rho]^* \tilde{\Delta}'$.

Definition 3.23. Let $\overset{K}{\Delta} \mathbf{X}$ be p -ECMSD and $\iota : \mathbf{B}_1 \implies \mathbf{X}$ a boundary map. For any $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, we define $K' := P_{\mathbf{B}_1}(\mathbb{A}^{(\infty)}) \cap {}^\rho K$, write

$$(\iota, \rho) : \overset{K'}{\Delta'} \mathbf{B}_1 \implies \overset{K}{\Delta} \mathbf{X},$$

and call this a boundary component of (or boundary morphism to) the p -ECMSD $\overset{K}{\Delta} \mathbf{X}$. Δ' is defined as $([\iota, \rho]^* \Delta)|_{\mathbf{B}_1}$, where restriction is characterized by

$$\Delta|_{\mathbf{B}_1}(\mathbb{D}_{\mathbf{B}_1}^0, P_{\mathbf{B}_2}, \rho_1) = \Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}_2}, \rho_1)$$

for all $\rho_1 \in P_{\mathbf{B}_1}(\mathbb{A}^{(\infty)})$, every boundary map $\mathbf{B}_2 \implies \mathbf{B}_1$ and every pair of connected components $\mathbb{D}_{\mathbf{X}}^0$ and $\mathbb{D}_{\mathbf{B}_1}^0$ such that $\mathbb{D}_{\mathbf{X}}^0 \hookrightarrow \mathbb{D}_{\mathbf{B}_1}^0 \hookrightarrow \mathbb{D}_{\mathbf{B}_2}$. Δ' in general inherits neither completeness nor finiteness. It is K' -admissible.

We call two boundary components

$$(\iota', \rho') : \overset{K'}{\Delta'} \mathbf{B}' \implies \overset{K}{\Delta} \mathbf{X}$$

and

$$(\iota'', \rho'') : \overset{K''}{\Delta''} \mathbf{B}'' \implies \overset{K}{\Delta} \mathbf{X}$$

equivalent, if (the images of) \mathbf{B}' and \mathbf{B}'' are conjugated via $\alpha \in P_{\mathbf{X}}(\mathbb{Q})$ and

$$\alpha \rho' \in \text{Stab}_{Q(\mathbb{Q})}(\mathbb{D}_{\mathbf{B}}) P_{\mathbf{B}}(\mathbb{A}^{(\infty)}) \rho'' K.$$

(Here Q is the parabolic defining \mathbf{B} .) In an equivalence class, we may assume $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty, p)})$.

Definition 3.24. Let ${}^K_{\Delta}\mathbf{X}$ be p -ECMSD. Define Δ^0 as the set of all $\sigma \in \Delta$, such that $\sigma \subset C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ for some improper rational boundary component $\mathbf{B} \implies \mathbf{X}$. We say that Δ is **concentrated in the unipotent fibre**, if $\Delta = \Delta^\circ$.

4. Symplectic, Hodge type, and Abelian type Shimura data

4.1. Let S be a scheme. Let L be a locally free sheaf on S with unimodular alternating form $\langle v, w \rangle = \Psi(v, w)$, i.e. satisfying $\langle v, v \rangle = 0$ and such that the induced homomorphism $L \rightarrow L^*$ is an isomorphism.

Let us assume first that L is *non-zero*. We then have the reductive group schemes $\mathrm{Sp}(L)$ (resp. $\mathrm{GSp}(L)$) over S , the **symplectic group**, resp. the **group of symplectic similitudes** of L .

There is an exact sequence

$$0 \longrightarrow \mathrm{Sp}(L) \longrightarrow \mathrm{GSp}(L) \xrightarrow{\lambda} \mathbb{G}_m \longrightarrow 0,$$

where λ is the similitude factor. Let ρ denote the standard representation of $\mathrm{GSp}(L)$ on L .

Let now $S = \mathrm{spec}(\mathbb{Z}_{(p)})$. We define p -integral pure Shimura data \mathbf{H}_g associated with L (it depends, up to isomorphism, only on the rank $2g$ of L) by $P_{\mathbf{H}_g} := \mathrm{GSp}(L)$ and $\mathbb{D}_{\mathbf{H}_g}$ to be the conjugacy class of morphisms $h : \mathbb{S} \rightarrow \mathrm{GSp}(L_{\mathbb{R}})$, such that they give pure Hodge structures of type $(-1, 0), (0, -1)$ on L and which are **polarized**, i.e. such that the form $\langle \cdot, h(i) \cdot \rangle$ is symmetric and (positive or negative) definite.

If L is the *zero* sheaf on S we define $\mathrm{Sp}(L) := 1$ and $\mathrm{GSp}(L) := \mathbb{G}_{m,S}$, and we let $\mathbb{D}_{\mathbf{H}_0}$ be the 2 point set of isomorphisms $\mathbb{Z} \rightarrow \mathbb{Z}(1)$ with the nontrivial action of $\mathbb{G}_m(\mathbb{R})$. This defines a (p -integral) Shimura datum \mathbf{H}_0 . We understand the morphism $\lambda : \mathrm{GSp}(0) \rightarrow \mathbb{G}_m$ be the identity.

4.2. Let L_0 be a locally free sheaf on S as before with unimodular symplectic form (possibly 0).

Let I be another locally free sheaf on S (also possibly 0). Let I^* be its dual. We define the semi-direct product

$$\mathrm{PSp}(L_0, I) := \mathbb{W}(L_0 \otimes I) \rtimes \mathrm{GSp}(L_0).$$

It is of type (P). Here $\mathrm{GSp}(L_0)$ acts via the product of the standard representation on L_0 with the trivial one on I . It acts on $L := L_0 \oplus I^*$ as follows: The action of GSp is given by the standard representation on L_0 and trivial action on I^* , $X = v' \otimes u' \in L_0 \otimes I$ acts, considered as element of the Lie algebra, as

$$X(v, u^*) = ((u^*u)v', 0)$$

and $\mathbb{W}(L_0 \otimes I)$ via the exponential $\exp(X)(v, u^*) = (v, u^*) + X(v, u^*)$, which in this case makes sense over any S . This is compatible with the structure of

semi-direct product. Let $S = \text{spec}(\mathbb{Z}_{(p)})$ from now on. The action fixes a weight filtration

$$W_i(L_0 \oplus I^*) := \begin{cases} 0 & i \leq -2 \\ L_0 & i = -1 \\ L_0 \oplus I^* & i \geq 0. \end{cases}$$

We have the unipotent extension $\mathbf{H}_{g_0}[0, I \otimes L_0]$ of the p -integral pure Shimura data \mathbf{H}_{g_0} . Its underlying P is $\text{PSp}(L_0, I)$. Its underlying \mathbb{D} (if $g_0 \neq 0$ may be identified with *the* conjugacy class of morphisms $h : \mathbb{S} \rightarrow \text{PSp}(L_0, \mathbb{R}, I_{\mathbb{R}})$, such that they give mixed Hodge structures of type $(-1, 0), (0, -1), (0, 0)$ with respect to the weight filtration W , such that on W_{-1} they are polarized. (We have $\text{PSp}(0, I) = \text{GSp}(0)$ and $\text{PSp}(L_0, 0) = \text{GSp}(L_0)$).

4.3. Let L_0 be a locally free sheaf on S with unimodular symplectic form as before (possibly 0). Consider the following extension of Abelian unipotent groups (**Heisenberg group**)

$$0 \longrightarrow \mathbb{W}((I \otimes I)^s) \longrightarrow \text{WSp} \longrightarrow \mathbb{W}(L_0 \otimes I) \longrightarrow 0$$

defined (if 2 is invertible in S) by the following group law

$$(u_1 u_2, v u_3)(u'_1 u'_2, v' u'_3) = (u_1 u_2 + u'_1 u'_2 + \frac{1}{2} \langle \mathbf{v}, \mathbf{v}' \rangle (\mathbf{u}_3 \mathbf{u}'_3 + \mathbf{u}'_3 \mathbf{u}_3), v u_3 + v' u'_3).$$

There is an action of $\text{GSp}(L_0)$ on WSp given by λ acting on $(I \otimes I)^s$ by scalars and standard representation tensored with the trivial one on $L_0 \otimes I$.

We define the semi-direct product

$$\text{USp}(L_0, I) := \text{WSp} \rtimes \text{GSp}(L_0).$$

It is again of type (P).

Denote $\mathfrak{I} := I \oplus I^*$. Choose on $L_0 \oplus \mathfrak{I}$ the symplectic form

$$\langle v_1, u_1, u_1^*; v_2, u_2, u_2^* \rangle := \langle v_1, v_2 \rangle + u_2^* u_1 - u_1^* u_2.$$

We define an action of $\text{USp}(L_0, I)$ on $L := L_0 \oplus \mathfrak{I}$ as follows: The action of $\text{GSp}(L_0)$ is given by the standard representation on L_0 , trivial representation on I^* and λ acting by scalars on I . $X = v' \otimes u' \in L_0 \otimes I$ acts, considered as element of the Lie algebra, as

$$X(v, u, u^*) = ((u^* u') v', \langle v', v \rangle u', 0),$$

and $X = u_1 \otimes u_2 \in I \otimes I$ acts by

$$X(v, u, u^*) = (0, (u^* u_1) u_2, 0),$$

and $\mathbb{W}(\dots)$ acts via the exponential $\exp(X)(v, u^*) = (v, u^*) + X(v, u^*) + \frac{1}{2} X^2(v, u^*)$. This is compatible with the group structure given above.

If 2 is not invertible in S , we assume that there is an isomorphism $L_0 \cong L_{00} \oplus L_{00}^*$, such that the alternating form is given by the standard one. In this case we let the groups $\mathbb{W}(L_{00} \otimes I)$, $\mathbb{W}(L_{00}^* \otimes I)$ and $\mathbb{W}((I \otimes I)^s)$ operate as above via exponential (it terminates after the second step). One checks that

$$\mathbb{W}((I \otimes I)^s) \times \mathbb{W}(L_{00} \otimes I) \times \mathbb{W}(L_{00}^* \otimes I) \rightarrow \mathrm{GSp}(L)$$

is a closed embedding onto a subgroup scheme. Explicitly the group law is given by

$$(X_1, X_2, X_3)(X'_1, X'_2, X'_3) = (X_1 + X'_1 + \langle \mathbf{X}_3, \mathbf{X}'_2 \rangle - \langle \mathbf{X}'_2, \mathbf{X}_3 \rangle, X_2 + X'_2, X_3 + X'_3).$$

This is in any case compatible with the structure of semi-direct product. Let $S = \mathrm{spec}(\mathbb{Z}_{(p)})$ from now on. The action fixes a weight filtration

$$W_i(L) := \begin{cases} 0 & i \leq -3, \\ I & i = -2, \\ L_0 \oplus I & i = -1, \\ L & i \geq 0. \end{cases}$$

We have the unipotent extension $\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]$ of the p -integral pure Shimura data \mathbf{H}_{g_0} . Its underlying P is $\mathrm{USp}(L_0, I)$. Its underlying \mathbb{D} (if $g_0 \neq 0$) may be identified with *the* conjugacy class of morphisms $h : \mathbb{S} \rightarrow \mathrm{USp}(L_{0, \mathbb{R}}, I_{\mathbb{R}})$, such that they give (polarized) mixed Hodge structures on $L_{\mathbb{R}}$ of type $(-1, -1)$, $(-1, 0)$, $(0, -1)$, $(0, 0)$ with respect to the weight filtration above.

The action of $\mathrm{USp}(L_0, I)$ on $L = L_0 \oplus \mathfrak{I}$ defined above induces a closed embedding

$$\mathrm{USp}(L_0, I) \hookrightarrow \mathrm{GSp}(L).$$

The image is precisely the subgroup scheme fixing the weight filtration and $I^* = \mathrm{gr}_0(L_0 \oplus \mathfrak{I})$ point-wise.

Definition 4.4. *For any saturated submodule $U' \subseteq (I \otimes I)^s$ we call the p -integral mixed Shimura data*

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0] / \mathbb{W}(U')$$

a Shimura datum of symplectic type.

Note that any of the p -integral mixed Shimura data in this section is of this form.

Proposition 4.5 ([18, Theorem 2.5.4]). *There is a bijection*

$$\{ \text{isotropic subspaces } I_{\mathbb{Q}} \text{ of } L_{\mathbb{Q}} \} \cong \{ \text{boundary components } \mathbf{B} \text{ of } \mathbf{H}_g \}.$$

Let $I_{\mathbb{Q}}$ be an isotropic subspace of dimension $g - g_0$, $I = I_{\mathbb{Z}_{(p)}}$ the associated saturated sublattice and $\mathbf{B} \implies \mathbf{H}_g$ the corresponding boundary component.

For any choice of splitting $L \cong L_0 \oplus \mathfrak{I}$ (where $\mathfrak{I} = I^* \oplus I$, as usual, with natural symplectic form), there is an isomorphism

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0] \cong \mathbf{B},$$

whose underlying morphism is given by the representation of $\mathrm{PSp}(L_0, I)$ on $L = L_0 \oplus I \oplus I^*$ given above (it depends on the choice of splitting).

Corollary 4.6. *Every boundary component of (p -integral) mixed Shimura data of symplectic type is again of symplectic type.*

4.7. We have quotient maps of mixed Shimura data.

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0] \rightarrow \mathbf{H}_{g_0}[0, I \otimes L_0] \rightarrow \mathbf{H}_{g_0}.$$

The first gives an isomorphism

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]/U \cong \mathbf{H}_{g_0}[0, I \otimes L_0]$$

(where $U = \mathbb{W}((I \otimes I)^s)$), and the second gives an isomorphism

$$\mathbf{H}_{g_0}[0, I \otimes L_0]/W \cong \mathbf{H}_{g_0},$$

(where $W = V = \mathbb{W}(I \otimes L_0)$).

Lemma 4.8. *Let S be a scheme with $H_{\text{et}}^1(S, \mathbb{G}_m) = 1$ and M a locally free sheaf on S .*

1. *A $(\mathrm{GSp}(L), L)$ -structure on M is a unimodular symplectic form on M (i.e. inducing an iso $M \cong M^*$) up to multiplication by $H^0(S, \mathcal{O}_S^*)$.*
2. *A $(\mathrm{PSp}(L_0, I), L)$ -structure on M is a saturated filtration*

$$W_0 = M \supset W_{-1} \supset W_{-2} = 0$$

with unimodular symplectic form on W_{-1} up to multiplication by $H^0(S, \mathcal{O}_S^)$ and an isomorphism $\rho : W_0/W_{-1} \cong I_S^*$.*

3. *A $(\mathrm{USp}(L_0, I), L)$ -structure on M is a unimodular symplectic form on M up to multiplication by $H^0(S, \mathcal{O}_S^*)$, a saturated filtration*

$$W_0 = M \supset W_{-1} \supset W_{-2} \supset W_{-3} = 0$$

such that W_{-2} is isotropic and $W_{-1} = (W_{-2})^\perp$ and an isomorphism $\rho : W_0/W_{-1} \cong I_S^$.*

Similar statements are true for a $(\mathrm{GSp}(L)(R), L_R)$ (resp. ...) structure on a local system.

Lemma 4.9. *Let S be a scheme and M a locally free sheaf on S . Let $G = \mathbb{W}(U_{\mathbb{Z}_{(p)}}) \rtimes \mathbb{G}_m$ acting on $\mathbb{Z}_{(p)} \oplus U_{\mathbb{Z}_{(p)}}^*$ as follows: \mathbb{G}_m acts by scalar multiplication on $\mathbb{Z}_{(p)}$ and $\mathbb{G}_a(U_{\mathbb{Z}_{(p)}})$ acts via $u(x, u^*) = (x - u^*u, u^*)$. Let S be a scheme over $\mathbb{Z}_{(p)}$ and M be a locally free sheaf on S .*

1. A $(G, \mathbb{Z}_{(p)} \oplus U_{\mathbb{Z}_{(p)}}^*)$ -structure on M is a saturated filtration

$$W_0 = M \supset W_{-1} = W_{-2} \supset W_{-3} = 0$$

with isomorphism

$$\rho : W_0/W_{-1} \cong U_S^*$$

and such that W_{-1} is locally free of rank 1.

4.10. Clearly, given two different representations L, L' of a group scheme G , there is a 1:1 “change of representation correspondence” between G -structures associated with those representations. For a translation of a certain such change of representation correspondence of $\mathrm{USp}(0, I)$ to 1-motives (cf. 7.9), we need a fairly explicit construction of this correspondence in terms of linear algebra.

Let S be a $\mathbb{Z}_{(p)}$ -scheme. Let $G = \mathrm{USp}(0, I)$ and L its standard representation. Let M be a locally free sheaf on S with (G, L) -structure. We may associate with it a $(G, \mathbb{Z}_{(p)} \oplus U_{\mathbb{Z}_{(p)}}^*)$ -structure M'' by the following (local) procedure: Define M' to be the quotient of $M \otimes I^*$ by the kernel of contraction $W_{-1} \otimes I^* \cong (I \otimes I^*)_S \rightarrow \mathcal{O}_S$ (the isomorphism is defined only up to scalars). Claim: The set $\Lambda^2 I^*$ has a unique lift to M' (under the map $M \otimes I^* \rightarrow (M/W_{-1}M) \otimes I^* = I^* \otimes I^*$) defined as follows: Let $i_1^* \wedge i_2^*$ be in $\Lambda^2 I^*$. We choose lifts $\tilde{i}_1^*, \tilde{i}_2^* \in M$ with the property that $\langle \tilde{i}_1^*, \tilde{i}_2^* \rangle = 0$. The projection of the element $\tilde{i}_1^* \otimes i_2^* - \tilde{i}_2^* \otimes i_1^*$ in M' is then well-defined (for this consider a difference $(\tilde{i}_1^* - \tilde{i}_1^*) \otimes i_2^* - (\tilde{i}_2^* - \tilde{i}_2^*) \otimes i_1^*$. Both summands lie already in $W_{-1}M \otimes I^*$ and, since $W_{-1}M$ is isotropic, its contraction may be computed by the expression: $\langle \tilde{i}_1^* - \tilde{i}_1^*, \tilde{i}_2^* \rangle - \langle \tilde{i}_2^* - \tilde{i}_2^*, \tilde{i}_1^* \rangle$. This is zero because of the choice of lifts.) We define $M'' = M'/\Lambda^2(I^*)$ by means of this lift. We get an inherited filtration:

$$0 \subseteq W_{-1}M'' \subseteq M''$$

such that $W_{-1}M'' \cong \mathcal{O}_S$ non-canonically and $M''/W_{-1}M'' \cong \mathrm{Sym}^2(I^*)$ canonically. That is, glueing these local constructions, we get a $(G, \mathbb{Z}_{(p)} \oplus U_{\mathbb{Z}_{(p)}}^*)$ -structure, where $U_{\mathbb{Z}_{(p)}} = (I \otimes I)^s$. The construction may be reversed.

Definition 4.11. Let \mathbf{X} be p -integral mixed Shimura data (p -MSD). \mathbf{X} is called of **Hodge type**, if there is an embedding into a p -integral mixed Shimura datum of symplectic type (see 4.4 for the notation)

$$\mathbf{X} \hookrightarrow \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U').$$

Also all ${}^K\mathbf{X}$ and ${}^K_{\Delta}\mathbf{X}$ are called of Hodge type.

A mixed Shimura datum \mathbf{X} is called of **Abelian type**, if there is a mixed Shimura datum of Hodge type \mathbf{X}' , a central isogeny $P_{\mathbf{X}'}^{\mathrm{der}} \rightarrow P_{\mathbf{X}}^{\mathrm{der}}$ which induces an isomorphism of mixed Shimura data $\mathbf{X}'^{\mathrm{ad}} \cong \mathbf{X}^{\mathrm{ad}}$.

Proposition 4.12. Let \mathbf{X} be p -integral mixed Shimura data and \mathbf{B} a boundary component of \mathbf{X} . If \mathbf{X} is of Hodge (resp. Abelian) type then \mathbf{B} is either.

Proof. An easy extension of [18, Theorem 2.6.3]. \square

5. Orthogonal Shimura data

5.1. Let L be a $\mathbb{Z}_{(p)}$ -lattice with unimodular quadratic form Q_L . We have the reductive group schemes $\mathrm{SO}(L)$ and $\mathrm{GSpin}(L)$. We let $\mathrm{SO}(L)$ act on L by its natural operation and $\mathrm{GSpin}(L)$ by $gv \mapsto gvg'$, where $'$ is the main involution of the Clifford algebra $C(L)$ of L . Note that these actions are *not* compatible with the projection $l : \mathrm{GSpin}(L) \rightarrow \mathrm{SO}(L)$. The group $\mathrm{GSpin}(L)$ is equipped with a morphism $\lambda : \mathrm{GSpin}(L) \rightarrow \mathbb{G}_m$ whose kernel is the Spin group.

For the rest of this section we assume $p \neq 2$. Then there exists an orthogonal basis $\{v_i\}$ of L , such that all $Q_L(v_i)$ are units. We have the basis $v_{i_1} \cdots v_{i_j}, i_1 < \cdots < i_j, j$ even of $C^+(L)$ and the basis $v_{i_1} \cdots v_{i_j}, i_1 < \cdots < i_j, j$ odd of $C^-(L)$. The trace of any basis element (acting by left multiplication on $C^+(L)$ or $C^-(L)$) except 1 is 0. The trace of 1 is 2^{m-1} in any case.

Lemma 5.2. *For an element $\delta \in C^+(L)^*$ with $\delta' = -\delta$, the form*

$$\langle x, y \rangle_\delta \mapsto \mathrm{tr}(x\delta y')$$

on $C^+(L)$ is symplectic, unimodular, and $\mathrm{Spin}(Q)$ -invariant (resp. $\mathrm{GSpin}(Q)$ -invariant up to scalar given by λ), where these groups act by left multiplication.

Proof. The form is invariant because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Since δ is invertible, nondegeneracy is equivalent to that of $\mathrm{tr}(xy')$ which is given by an invertible diagonal matrix with respect to the basis chosen above. The form is symplectic because tr is invariant under $'$ and $\delta' = -\delta$. \square

5.3. Suppose again, $L_{\mathbb{Z}_{(p)}}$ is a lattice with unimodular quadratic form. Suppose that $L_{\mathbb{R}}$ has signature $(m-2, 2)$, $m \geq 3$. A polarized Hodge structure of type $(-2, 0), (-1, -1), (0, -2)$ on $L_{\mathbb{C}}$ with $\dim(L^{-2,0}) = 1$ is determined by an isotropic subspace $L^{-2,0}$, satisfying $\langle z, \bar{z} \rangle < 0$ for non-zero $z \in L^{-2,0}$. Then the other spaces in the Hodge decomposition are determined by $L^{0,-2} = \overline{L^{-2,0}}$ and $L^{-1,-1} = (L^{-2,0} + L^{0,-2})^\perp$. We define $\mathbb{D}_{\mathbf{S}(L)} = \mathbb{D}_{\mathbf{O}(L)}$ to be the set of these Hodge structures. It is identified with

$$\{\langle z \rangle \in \mathbb{P}(L_{\mathbb{C}}) \mid \langle z, z \rangle = 0, \langle z, \bar{z} \rangle < 0\}.$$

It is equipped with a complex structure coming from $\mathbb{P}(L_{\mathbb{C}})$.

$\mathbb{D}_{\mathbf{S}}$ has 2 connected components. There is a 2:1 map

$$\begin{aligned} \mathbb{D}_{\mathbf{S}(L)} &\rightarrow \mathrm{Grass}^-(L_{\mathbb{R}}) \\ \langle z \rangle &\mapsto D_{\langle z \rangle} := (\langle z, \bar{z} \rangle)^{\mathrm{Gal}(\mathbb{C}|\mathbb{R})} \end{aligned}$$

onto the Grassmannian Grass^- of negative definite subspaces of $L_{\mathbb{R}}$. It has a section determined by a choice of \mathbb{C} -orientation⁴ on any of the subspaces D . The images of these sections are the 2 connected components of $\mathbb{D}_{\mathbf{S}}$.

⁴i.e. the choice of a \mathbb{R} -linear isometry $D \cong \mathbb{C}$

It will be convenient to fix an orientation on some (hence on all) of the negative definite subspaces D . This identifies $\pi_0(\mathbb{D}_{\mathbf{S}(L)})$ with $\mathbb{D}_{\mathbf{H}_0} = \text{Hom}(\mathbb{Z}, \mathbb{Z}(1))$ and induces a morphism of p -MSD

$$\mathbf{S}(L) \rightarrow \mathbf{H}_0.$$

The space $D_{<z>}$ has an orthonormal (with respect to Q_L) basis given by

$$x_1 = \frac{z + \bar{z}}{\sqrt{-\langle z, \bar{z} \rangle}}, \quad x_2 = \pm \frac{z - \bar{z}}{\sqrt{\langle z, \bar{z} \rangle}}. \quad (2)$$

Lemma 5.4. $\mathbf{S}(L)$ is p -integral pure Shimura data.

If L has signature $(0, 2)$, GSpin does not operate at all on $\mathbb{D}_{\mathbf{S}}$, if the latter is defined as above (i.e. consisting of 2 points). Therefore, in this case we get 2 different p -integral Shimura data $\mathbf{S}(L)_{\pm}$. If L' is a saturated sublattice of L of signature $(m - n - 2, 2)$, $0 \leq n \leq m - 2$, it induces an embedding of p -integral Shimura data

$$\mathbf{S}(L') \hookrightarrow \mathbf{S}(L).$$

For any chosen $\delta \in C^+(L)^*$ with $\delta' = -\delta$ there is an embedding

$$\mathbf{S}(L) \hookrightarrow \mathbf{H}(C^+(L), \langle \cdot, \cdot \rangle_{\delta})$$

and accordingly $\mathbf{S}(L)$ is of Hodge type. $\mathbf{O}(L) := \mathbf{S}(L)/\mathbb{G}_m$ is of Abelian type.

Proof. To each $<z> \in \mathbb{D}_{\mathbf{S}(L)}$ the associated morphism h factors through $\text{GSpin}(L_{\mathbb{R}})$. It is explicitly given by

$$\begin{aligned} \mathbb{S} &\rightarrow C^+(L, \mathbb{R}) \\ w = a + bi &\mapsto a + bi \frac{z\bar{z} - \bar{z}z}{\langle z, \bar{z} \rangle} = a + bx_1x_2 \end{aligned}$$

for any choice of $i \in \mathbb{C}$ (note: x_2 depends on the choice of $i \in \mathbb{C}$ as well). It even defines a field isomorphism $\mathbb{C} \cong C^+(D_{<z>}, \mathbb{R})$. It acts on $L^{-2,0}$ by w^2 , on $L^{0,-2}$ by \bar{w}^2 and on $L^{-1,-1}$ by $w\bar{w}$. The second statement is true because the inclusion $C^+(L') \subset C^+(L)$ induces a closed embedding $\text{GSpin}(L') \hookrightarrow \text{GSpin}(L)$, and any morphism h of the first datum defines a morphism h for the second datum of the same type.

For the property of Hodge type, fix a positive definite $\mathbb{Z}_{(p)}$ -sublattice D_0 which is a direct summand and an orthonormal basis x_1, x_2 (hence an induced orientation of $D_{0,\mathbb{R}}$). Consider the element $\delta := x_1x_2$. It satisfies the requirements of Lemma 5.2.

Let $<z>$ be a point in $\mathbb{D}_{\mathbf{S}(L)}$. $P_z^{-1,0} := \frac{z\bar{z}}{\langle z, \bar{z} \rangle}$ resp. its complex conjugate, satisfy $P_z^{-1,0} + P_z^{0,-1} = \text{id}$, $(P_z^{i,j})^2 = P_z^{i,j}$, and on $P_z^{i,j} C^+(L_{\mathbb{C}})$ the morphism h acts as $w^{-i}\bar{w}^{-j}$. Furthermore, for z chosen, such that $D_z = D_0$, the form $\langle \cdot, x_1x_2 \rangle_{\delta} = \langle \cdot, h(i)\cdot \rangle_{\delta} = \text{tr}(x\delta y'\delta') = -\text{tr}(x\delta y'\delta)$ is symmetric and definite. (Here $i \in \mathbb{C}$ is the root of -1 determined by the orientation $x_1 \wedge x_2$ of $D_0 = D_z$, see (2)). The substitution $x \mapsto gx$, $y \mapsto gy$ for any g in the spin group

changes x_1x_2 in $g(x_1)g(x_2)$ but does not affect the properties of symmetry and definiteness. The sign of definiteness, however, is reflected by the chosen orientation of D_0 . Hence the map $\langle z \rangle \mapsto P_z^{-1,0} C^+(L_{\mathbb{C}})$ may be seen as a map $\mathbb{D}_{\mathbf{S}(L)} \rightarrow \mathbb{D}_{\mathbf{H}(C^+(L), \langle \cdot, \cdot \rangle_{\delta})}$. Together with the closed embedding

$$\mathrm{GSpin}(L) \hookrightarrow \mathrm{GSp}(C^+(L), \langle \cdot, \cdot \rangle_{\delta}),$$

it induces an embedding $\mathbf{S}(L) \hookrightarrow \mathbf{H}(C^+(L), \langle \cdot, \cdot \rangle_{\delta})$. $\mathbf{O}(L)$ is, as quotient of $\mathbf{S}(L)$, clearly of Abelian type. \square

We have the following proposition (similar to 4.5):

Proposition 5.5. *There is a bijection*

$$\{ \text{isotropic subspaces } I_{\mathbb{Q}} \text{ of } L_{\mathbb{Q}} \} \cong \{ \text{boundary components } \mathbf{B} \text{ of } \mathbf{S}(L) \text{ or } \mathbf{O}(L) \}.$$

Let $I_{\mathbb{Q}}$ be an isotropic subspace, $I = I_{\mathbb{Z}_{(p)}}$ the associated saturated sublattice and \mathbf{B} the corresponding boundary component.

For any choice of splitting $L = I \oplus I^* \perp L_0$, there is an isomorphism (in each case, $\mathbf{S}(L)$ or $\mathbf{O}(L)$):

$$\mathbf{B} \cong \begin{cases} \mathbf{H}_0[I \otimes (I^{\perp}/I), 0] & \text{if } \dim(I) = 1, \\ \mathbf{H}_1(I)[\Lambda^2 I, I \otimes (I^{\perp}/I)] & \text{if } \dim(I) = 2, \end{cases}$$

where in the first case \mathbb{G}_m acts on $I \otimes (I^{\perp}/I)$ by scalars and $\mathrm{GL}(I)$ in the second case acts trivial on I^{\perp}/I (see the definition of unipotent extension 3.10). The isomorphisms depend in addition on a common choice of orientation on the maximal negative definite subspaces of $L_{\mathbb{R}}$.

Proof. According to Proposition 3.14 we have to classify admissible parabolic subgroups of $\mathrm{SO}(L)$ or $\mathrm{GSpin}(L)$ (this amounts to the same). We could also restrict ourselves to $\mathrm{GSpin}(L_{\mathbb{Q}})$ because of the projectivity of \mathcal{PAR} (2.8).

So let Q be any proper *admissible* parabolic 3.11 of $\mathrm{GSpin}(L)$. It is well-known that Q corresponds to a saturated filtration

$$0 \subset I \subseteq I^{\perp} \subset L,$$

with I isotropic, whose stabilizer it is. We will now determine the boundary component in the sense of 3.15 associated with Q . First note that we can identify $\mathrm{Lie}(\mathrm{SO}) = \mathrm{Lie}(\mathrm{Spin})$ with $\Lambda^2 L$ via the action:

$$(v \wedge w)x := \langle w, x \rangle v - \langle v, x \rangle w.$$

The filtration induces the following saturated filtration

$$0 \subseteq \Lambda^2 I \subseteq I \wedge I^{\perp}$$

consisting of those elements in $\Lambda^2 L$ shifting the filtration by at least -2, resp. -1. We have $\mathrm{Lie}(W_Q) = I \wedge I^{\perp}$, where W_Q is the unipotent radical of Q (exists for parabolics, e.g. by the fact that they are of type (P), cf. 2.6).

5.6. Let L be a unimodular quadratic form over a discrete valuation ring R with 2 invertible. Decompose $L_R = H_{1,R} \perp \cdots \perp H_{k,R} \perp L'_R$, where L' is anisotropic over R . We get an R -split torus of GSpin as follows: Let z_i, z'_i be an isotropic basis of $H_{i,R}$ with $\langle z_i, z'_i \rangle = 1$. We get a 1-parameter subgroup T_i consisting of

$$t_i(\alpha) := \alpha z_i z'_i + z'_i z_i = 1 + (\alpha - 1) z_i z'_i$$

in $\mathrm{GSpin}(L)$. l induces an isomorphism of it with its image in $\mathrm{SO}(L)$. The product of all T_i is direct and yields a split torus T of $\mathrm{GSpin}(L)$. The product of it with the center of GSpin is a maximal R -split torus. l induces again an *isomorphism* of T with $l(T)$ which is a *maximal* R -split torus of SO . $t_i(\alpha)$ acts, via the standard representation of $\mathrm{GSpin}(L)$ by $z_i \mapsto \alpha^2 z_i$, $z'_i \mapsto z'_i$ and $v \mapsto \alpha v$ for $v \in \langle z, z' \rangle^\perp$. $l(t_i(\alpha))$ acts, via the standard representation of $\mathrm{SO}(L)$ by $z_i \mapsto \alpha z_i$, $z'_i \mapsto \alpha^{-1} z'_i$ and $v \mapsto v$ for $v \in \langle z, z' \rangle^\perp$.

5.7. Let $L_{\mathbb{R}}$ be a quadratic space over \mathbb{R} with signature $(m-2, 2)$. Decompose $L_{\mathbb{R}} = H_{1,\mathbb{R}} \perp H_{2,\mathbb{R}} \perp L'_{\mathbb{R}}$, where $H_{i,\mathbb{R}}$ are hyperbolic planes. A maximal \mathbb{R} -split torus of $\mathrm{SO}(L_{\mathbb{R}})$ is $l(T_1 \cdot T_2) \cong \mathbb{G}_m^2$. The long roots (3.13) are in this case: $\alpha_1 = (1, -1)$ and $\alpha_2 = (1, 1)$ in $\mathbb{Z}^2 = \mathrm{Hom}(l(T_1 T_2), \mathbb{G}_m)$. The weight morphism $\mu := \mu_x = h_x \circ w$ for any $x \in \mathbb{D}_{\mathbf{S}(L)}$ is given by the natural inclusion $\mathbb{G}_m \hookrightarrow \mathrm{GSpin}$. The homomorphism λ satisfying the criterion in 3.13, in particular mapping to $\mathrm{Spin}(L)$, and its product with μ is given in the following table:

| subset of long roots | λ | $\lambda \cdot \mu$ |
|--------------------------|--------------|---------------------|
| \emptyset | $(0; 0, 0)$ | $(1; 0, 0)$ |
| $\{\alpha_1\}$ | $(0; 1, -1)$ | $(1; 1, -1)$ |
| $\{\alpha_2\}$ | $(-1; 1, 1)$ | $(0; 1, 1)$ |
| $\{\alpha_1, \alpha_2\}$ | $(-1; 2, 0)$ | $(0; 2, 0)$ |

Here $(l; m, n)$ denotes the cocharacter of GSpin : $\alpha \mapsto \alpha^l t_1(\alpha^m) t_2(\alpha^n)$.

5.8. Case: I 1 dimensional (boundary point).

Choose a splitting $L = I \oplus I^* \perp L_0$. This is possible because the discriminant is a unit at p . We may assume $H_{1,\mathbb{R}} = I_{\mathbb{R}} \oplus I_{\mathbb{R}}^*$. Only the last cocharacter of 5.7 is defined over \mathbb{Q} and yields a proper parabolic subgroup.

The morphism $\lambda \cdot \mu$ which is a weight morphism for the boundary component has the following eigenspaces:

$$L^{-4} = I \quad L^{-2} = L_0 \quad L^0 = I^*.$$

We get a Levi decomposition

$$Q = \mathbb{W}(I \otimes (I^\perp/I)) \rtimes G,$$

where $\mathbb{W}(I \otimes (I^\perp/I))$ is the unipotent radical and acts via the exponential (note $I \wedge I^\perp \cong I \otimes (I^\perp/I)$ in this case).

We interpreted the set $\mathbb{D}_{\mathbf{S}(L)}$ as the set of Hodge structures of weight 2 on $L_{\mathbb{R}}$ with $L_{-2,0}$ isotropic, and therefore $h(\mathbb{D}_{\mathbf{B}})$ can be interpreted as an orbit in the

set of mixed Hodge structures with respect to the weight filtration

$$W_i(L) = \begin{cases} 0 & i \leq -5 \\ I & i = -4, -3 \\ I^\perp & i = -2, -1 \\ L & i \geq 0 \end{cases}$$

containing the same Hodge filtrations as above.

Since the Hodge filtration type stays the same, the induced Hodge structures on $\text{gr}_W(L)$ have to be trivial by weight reasons. Also, we see that a mixed Hodge structure is given by

$$0 \subseteq F^0 = I_{\mathbb{C}}^* \subseteq F^{-1} = (I^*)^\perp \subseteq L_{\mathbb{C}}. \quad (3)$$

The associated morphism $h_0 : \mathbb{S}_{\mathbb{C}} \rightarrow \text{GSpin}(L_{\mathbb{C}})$ is given by $z \mapsto t_1(z\bar{z})$. Its weight morphism is precisely the morphism $\lambda \cdot \mu$ constructed above. The product $\mathbb{W}(I \otimes (I^\perp/I))T_1$ is already a closed normal subgroup scheme of Q , hence equal to $P_{\mathbf{B}}$. It is the same as the subgroup scheme underlying $\mathbf{H}_0[I \otimes (I^\perp/I), 0]$.

Consider

$$\pi_0(\mathbb{D}_{\mathbf{S}}) \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}}).$$

$T_1(\mathbb{R})$ acts by the nontrivial operation on the set $\pi_0(\mathbb{D}_{\mathbf{S}})$.

Choose vectors z, z' spanning L and L^* respectively. Via the isomorphism $L_0 \rightarrow I \otimes I^\perp/I$, $k \mapsto z \otimes k$, a vector $k \in L_{0, \mathbb{C}}$ maps the mixed Hodge structure (3) above to the one determined by

$$F^0 = \langle z' + \langle z, z' \rangle k - \left(\langle k, z' \rangle + \frac{1}{2} \langle z, z' \rangle \langle k, k \rangle \right) z \rangle.$$

If $\Im(k) > 0$, this is the filtration associated with a Hodge structure in $\mathbb{D}_{\mathbf{S}(L)}$. Hence by Definition 3.15 the $T_1(\mathbb{R})L_{0, \mathbb{C}}$ -orbit $\mathbb{D}_{\mathbf{B}}$ in $\pi_0(\mathbb{D}_{\mathbf{S}}) \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}})$ is equal to $\pi_0(\mathbb{D}_{\mathbf{S}}) \times L_{0, \mathbb{C}} \cdot h_0$.

This is (see Definition 3.10) isomorphic to

$$\mathbb{D}_{\mathbf{H}_0[I \otimes (I^\perp/I), 0]},$$

where $\pi_0(\mathbb{D}_{\mathbf{S}})$ can be identified (after choice of a common orientation of the negative definite subspaces) with the set of isomorphisms $\mathbb{Z} \cong \mathbb{Z}(1)$ (choice of root of -1). Combining this with the isomorphism of group schemes above, we get an isomorphism

$$\mathbf{B} \cong \mathbf{H}_0[I \otimes (I^\perp/I), 0],$$

as required.

5.9. Case: I 2 dimensional (boundary curve).

We may now assume that the whole torus T_1T_2 is defined over \mathbb{Q} . We get an associated maximal parabolic by considering the two parabolics associated with

the morphisms $(1; 1, -1)$ and $(0; 1, -1)$. The morphism $\lambda \cdot \mu$ which is a weight morphism for the boundary component has the following eigenspaces:

$$L^{-3} = \langle z, w' \rangle \quad L^{-2} = L_0 \quad L^{-1} = \langle z', w \rangle$$

In the other case the morphism $\lambda \cdot \mu$ which is a weight morphism for the boundary component has the following eigenspaces:

$$L^{-3} = \langle z, w \rangle \quad L^{-2} = L_0 \quad L^{-1} = \langle z', w' \rangle$$

If $m = 4$ these are *not* conjugated, in accordance with the fact that $\text{Spin}(L)$ is not simple.

We get

$$W_i(L) = \begin{cases} 0 & i \leq -4 \\ I & i = -3 \\ I^\perp & i = -2 \\ L & i \geq -1. \end{cases}$$

For any isotropic $\langle z \rangle \subset I_{\mathbb{C}}$ and corresponding $\langle z' \rangle = \langle z \rangle^{\perp, I_{\mathbb{C}}^*}$, we get a filtration

$$0 \subseteq F^0 = \langle z' \rangle \subseteq F^{-1} = \langle z' \rangle^\perp \subseteq L_{\mathbb{C}}. \quad (4)$$

If $F^0 = \langle z \rangle \subset I_{\mathbb{C}}$ is a Hodge structure on $I_{\mathbb{C}}$, that is, if $\langle \bar{z} \rangle \neq \langle z \rangle$ then the filtration (4) defines a mixed Hodge structure on $L_{\mathbb{C}}$, together with the weight filtration above.

We have an exact sequence

$$0 \longrightarrow \Lambda^2(I) \longrightarrow I \wedge I^\perp \longrightarrow I \otimes (I^\perp / I) \longrightarrow 0$$

where $\Lambda^2(I)$ are the elements shifting the filtration by 2 and a corresponding central extension

$$0 \longrightarrow U_{\mathbf{B}} := \mathbb{W}(\Lambda^2(I)) \longrightarrow W_{\mathbf{B}} \longrightarrow V_{\mathbf{B}} := \mathbb{W}(I \otimes (I^\perp / I)) \longrightarrow 0$$

If we choose, in addition to our splitting, a basis z, w (as before) determining $I \cong (\mathbb{Z}_{(p)})^2$ and dual basis $I^* \cong (\mathbb{Z}_{(p)})^2$, we get an obvious isomorphism $\det : \Lambda^2(I) \cong \mathbb{Z}_{(p)}$ and an isomorphism $(I \otimes (I^\perp / I)) \cong L_0^2$. An element of $W_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$ may now be written as (u, k_1, k_2) with $u \in \mathbb{C}$ and $k_i \in L_{0, \mathbb{R}}$.

It sends the Hodge filtration (4) determined by $z = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}_I$ explicitly to the following one:

$$F^0 = \langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{I^*} + u \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}_I + \alpha k_1 + \beta k_2 - \begin{pmatrix} \alpha Q(k_1) \\ \beta Q(k_2) + \alpha \langle k_1, k_2 \rangle \end{pmatrix}_I \rangle.$$

(these again contain all filtrations of Hodge structures determined by elements in $h(\mathbb{D}_{\mathbf{S}(L)})$.)

We also get a morphism $\mathrm{SL}(I) \rightarrow \mathrm{SO}(L)$, letting a matrix act on I by the standard representation, on I^* by its contragredient and on L_0 trivially. It lifts to a morphism $\mathrm{SL}(I) \rightarrow \mathrm{Spin}(L) \subset \mathrm{GSpin}(L)$ because $\mathrm{SL}(I)$ is algebraically simply-connected. Note that $t_1(-1)t_2(-1)$ acts the same way than -1 under the above lift. Hence the morphism extends to a morphism $\rho : \mathrm{GL}(I) \rightarrow \mathrm{GSpin}(L)$ mapping the diagonal torus to T_1T_2 . It hence acts via the standard representation of GSpin as follows on L : By the standard representation times determinant on I , by the contragredient times determinant on I^* and by the determinant on L_0 . The mixed Hodge structure determined by (4) is hence given by the morphism $\mathbb{S}_{\mathbb{C}} \rightarrow \mathrm{GSpin}(L_{\mathbb{C}})$ obtained composing the morphism $h_0 : \mathbb{S} \rightarrow \mathrm{GL}(I_{\mathbb{R}})$ associated with the Hodge structure $F^0 = \langle z \rangle \subseteq I_{\mathbb{C}}$ with ρ . Its weight morphism is precisely the morphism $\lambda \cdot \mu$ constructed above. The product $W_{\mathbf{B}} \mathrm{GL}(L)$ is already a closed normal subgroup scheme of Q , hence equal to $P_{\mathbf{B}}$.

Consider

$$\pi_0(\mathbb{D}_{\mathbf{S}}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}}).$$

The stabilizer in $\mathrm{GL}(L_{\mathbb{R}})$ of h_0 is connected (in real topology) and hence acts trivially on the set $\pi_0(\mathbb{D}_{\mathbf{S}})$. Hence by Definition 3.15 the $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$ -orbit $\mathbb{D}_{\mathbf{B}}$ in $\pi_0(\mathbb{D}_{\mathbf{S}}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}})$ is isomorphic to its projection to $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B}, \mathbb{C}})$. This is (see Definition 3.10) isomorphic to

$$\mathbb{D}_{\mathbf{H}_1(I)}[\Lambda^2 I, I \otimes (I^{\perp}/I)].$$

Combining this with the isomorphism of group schemes above, we get an isomorphism:

$$\mathbf{B} \cong \mathbf{H}_1(I)[\Lambda^2 I, I \otimes (I^{\perp}/I)]$$

as required. \square

6. Integral models of Shimura varieties and their toroidal compactifications

6.1. To prove our main theorems about the arithmetic volume of Shimura varieties of orthogonal type, we need to assume that the following statements on canonical integral models of them and their toroidal compactifications hold true. We treat exclusively the case of good reduction. In this case the theorems have been proven under a certain hypothesis (6.6) in the thesis of the author [18].

Definition/Theorem 6.2. *Let \mathbf{X} be p -integral mixed Shimura data. With $x \in \mathbb{D}_{\mathbf{X}}$ there is associated $h_x : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbf{X}, \mathbb{C}}$, hence a morphism $u_x : \mathbb{G}_{m, \mathbb{C}} \rightarrow P_{\mathbf{X}, \mathbb{C}}$ via composition with the inclusion of the first factor into $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$. The conjugacy class M of these morphisms is defined over a number field $E_{\mathbf{X}}$, called the **reflex field**, unramified over p . We call the localization of its ring of integers $\mathcal{O} := \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ at p the **reflex ring**. It is a semi-local ring. The closure \overline{M} in $\underline{\mathrm{Hom}}(\mathbb{G}_{m, \mathcal{O}}, P_{\mathbf{X}, \mathcal{O}})$ maps surjectively onto $\mathrm{spec}(\mathcal{O})$.*

We consider the 2-category of Deligne-Mumford stacks over reflex rings as the 2-category of morphisms $(\pi : S \rightarrow \text{spec}(\mathcal{O}))$, where $\mathcal{O} \subset \mathbb{C}$ is the localization of a ring of integers at $\mathbb{Z} \setminus (p)$ and $\pi : S \rightarrow \text{spec}(\mathcal{O})$ is a morphism of Deligne-Mumford stacks. Morphisms are commutative diagrams where the map $\text{spec}(\mathcal{O}) \rightarrow \text{spec}(\mathcal{O}')$ has to come from the inclusion.

6.3. We begin by describing the functorial theory of (uncompactified) integral models of mixed Shimura varieties in the case of good reduction. The Main Theorem is essentially due to Kisin [19, 20] or Vasiu [39] in the pure case.

Definition 6.4. Let \mathcal{O} be a discrete valuation ring with fraction field F . Let a **test scheme** S over \mathcal{O} be as in [33, Def. 3.5], i.e. S has a cover by open affines $\text{spec}(A)$, such that there exist rings $\mathcal{O} \subseteq \mathcal{O}' \subseteq A_0 \subseteq A$, where

- $\mathcal{O} \subseteq \mathcal{O}'$ is a faithfully flat and unramified extension of d.v.r. with $\mathcal{O}'/(\pi)$ separable over $\mathcal{O}/(\pi)$.
- A_0 is a smooth \mathcal{O}' -algebra.
- $A_0 \subseteq A_1 \subseteq \cdots \subseteq A$ is a countable union of etale extensions.

Main Theorem 6.5. Let $p \neq 2$ be a prime.

There is a unique (up to unique isomorphism) pseudo-functor M from the category of p -EMSD ${}^K\mathbf{X}$ of Abelian type to the 2-category of smooth Deligne-Mumford stacks over reflex rings:

$${}^K\mathbf{X} \mapsto (M({}^K\mathbf{X}) \rightarrow \text{spec}(\mathcal{O}_{\mathbf{X}}))$$

with a pseudo-natural isomorphism:

$$[P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)] \cong (M({}^K\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathbb{C}))^{an},$$

satisfying the following properties:

1. If $(\gamma, \rho) : {}^{K'}\mathbf{Y} \rightarrow {}^K\mathbf{X}$ is an embedding,

$$M(\gamma, \rho) \times \pi_{\mathbf{Y}} : M({}^{K'}\mathbf{Y}) \rightarrow M({}^K\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathcal{O}_{\mathbf{Y}})$$

is a normalization followed by a closed embedding. If (γ, ρ) is a Hecke operator, then $M(\gamma, \rho) \times \pi_{\mathbf{Y}}$ is etale and finite.

2. (rational canonicity) For a pure Shimura datum \mathbf{Y} , where $P_{\mathbf{Y}}$ is a torus, the composition of the reciprocity isomorphism (normalized as in [36, 11.3])

$$\text{Gal}(\overline{E}|E)^{ab} \cong \pi_0(T_E(\mathbb{Q}) \backslash T_E(\mathbb{A})),$$

(where $T_E = \text{Res}_{\mathbb{Q}}^E(\mathbb{G}_m)$) with π_0 of the reflex norm [36, 11.4]

$$\pi_0(T_E(\mathbb{Q}) \backslash T_E(\mathbb{A})) \rightarrow \pi_0(P_{\mathbf{Y}}(\mathbb{Q}) \backslash P_{\mathbf{Y}}(\mathbb{A}))$$

defines by means of the natural action of $\pi_0(P_{\mathbf{Y}}(\mathbb{Q}) \backslash P_{\mathbf{Y}}(\mathbb{A}))$ on $M({}^K\mathbf{Y})(\mathbb{C})$ the rational model $M({}^K\mathbf{Y})_E$.

3. (integral canonicity) The projective limit

$$M^p(\mathbf{X}) := \varprojlim_K M(K\mathbf{X}),$$

where the limit is taken over all admissible compact open subgroups, satisfies the following property:

For any prime $\wp|p$ of $\mathcal{O}_{\mathbf{X}}$, each test scheme (6.4) T over $S = \text{spec}(\mathcal{O}_{\mathbf{X},\wp})$, and a given morphism

$$\alpha_E : T \times_S Q \rightarrow M^p(\mathbf{X}) \times_S Q,$$

where $Q = \text{spec}(E_{\mathbf{X}})$ is the generic fibre of S , there exists a uniquely determined morphism

$$\alpha : T \rightarrow M^p(\mathbf{X})$$

such that $\alpha_E = \alpha \times_S Q$.

4. $M(K\mathbf{X})$ is a quasi-projective scheme, if K is neat.

For $p = 2$, it may be necessary to change the notion of ‘integral canonicity’ as formulated above, cf. [33, 3.4].

The following Main Theorem 6.9 on canonical models of *toroidal compactifications* was only proven in [18] under the following hypothesis, which remains unproven:

Conjecture 6.6. *Let \mathbf{X} be pure p -integral Shimura data. Let any embedding ${}^K_{\Delta}\mathbf{X} \hookrightarrow {}^{K'}_{\Delta'}\mathbf{X}'$ in a datum of symplectic type $\mathbf{X}' = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$ be given, such that Δ and Δ' are smooth projective and K, K' are neat. Consider the toroidal compactification $M({}^{K'}_{\Delta'}\mathbf{X}')$ over $\text{spec}(\mathbb{Z}_{(p)})$, constructed in [12], and let D be its boundary divisor. For every prime $\wp|p$ of $\mathcal{O}_{\mathbf{X}}$ we have the following:*

Let $S = \text{spec}(\mathcal{O}_{\mathbf{X},\wp})$ and \mathcal{M}' be the Zariski closure of (the rational canonical model of) $P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)$ in $M({}^{K'}_{\Delta'}\mathbf{X}')_S$.

No connected component of $N(\mathcal{M}') \cap D_S$ lies entirely in the fibre above \wp .

6.7. The toroidal compactifications will have a stratification, which consists of mixed Shimura varieties itself. For this, we have to set up some notation (cf. [36, 7.3ff]). Let $(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \rightrightarrows {}^K_{\Delta}\mathbf{X}$ be a boundary component. For each cone $\sigma \in \Delta'$ with $\sigma \subset C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho'$ for some ρ' in the equivalence class of ρ , we define a p -ECMSD ${}^{K_{[\sigma]}}_{\Delta_{[\sigma]}}\mathbf{B}_{[\sigma]}$ as follows: $\mathbf{B}_{[\sigma]}$ is defined by

$$\begin{aligned} P_{\mathbf{B}_{[\sigma]}} &:= P_{\mathbf{B}} / \langle \sigma \rangle, \\ \mathbb{D}_{\mathbf{B}_{[\sigma]}} &:= P_{\mathbf{B}_{[\sigma]}}(\mathbb{R})U_{\mathbf{B}_{[\sigma]}}(\mathbb{C}) - \text{orbit generated by } (\mathbb{D}_{\mathbf{X}}^0 / \langle \sigma \rangle (\mathbb{C})) \times \{\sigma P_{\mathbf{X}}(\mathbb{A}^{(\infty)})\} \\ &\quad \text{in } (\mathbb{D}_{\mathbf{X}}^0 / \langle \sigma \rangle (\mathbb{C})) \times \{[\sigma]/P_{\mathbf{X}}(\mathbb{A}^{(\infty)})\} \text{ for } \sigma \in [\sigma] \cap \Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, 1). \end{aligned}$$

$K_{[\sigma]}$ is the image of K' under the projection $P_{\mathbf{B}} \rightarrow P_{\mathbf{B}_{[\sigma]}}$. $\Delta_{[\sigma]}$ is defined by

$$\Delta_{[\sigma]}(\mathbb{D}_{\mathbf{B}_{[\sigma]}}^0, P_{\mathbf{B}_{[\sigma]}}, \bar{\rho}) := \{\tau \bmod \langle \sigma \rangle \mid \tau \in \Delta(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}, \rho) \text{ such that } \sigma \text{ is a face of } \tau\}.$$

It is $K_{[\sigma]}$ -admissible.
Furthermore, we define

$$\Gamma := (\text{Stab}_{Q(\mathbb{Q})}(\mathbb{D}_{\mathbf{B}}) \cap (P_{\mathbf{B}}(\mathbb{A}^{(\infty)})^{(\rho K)})) / P_{\mathbf{B}}(\mathbb{Q}),$$

where Q is the parabolic describing \mathbf{B} .

$\Sigma(\iota, \rho)$ is defined to be the set of equivalence classes of the action of Γ (defined above) on those $[\sigma] \in P_{\mathbf{B}}(\mathbb{Q}) \backslash (\Delta')^0 / P_{\mathbf{B}}(\mathbb{A}^{(\infty)})$, which satisfy $\sigma \subset C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho'$ for some $\mathbb{D}_{\mathbf{X}}^0$.

6.8. Recall from 3.9: Any morphism

$$\mathbf{X} \rightarrow \mathbf{X}/U$$

is a torsor under the *group object*

$$\mathbf{X}/W_{\mathbf{X}}[U, 0] \rightarrow \mathbf{X}/W_{\mathbf{X}}$$

(cf. also [36, 6.6–6.8]). The corresponding map of Shimura varieties (assume all K 's in the sequel to be neat):

$$M^K(\mathbf{X}/W_{\mathbf{X}}[U, 0]) \rightarrow M^{K/U}(\mathbf{X}/W_{\mathbf{X}})$$

is a split torus with character group (in general non canonically) isomorphic to $U_{\mathbb{Q}} \cap K$. If Δ is a K -admissible polyhedral cone decomposition for \mathbf{X} , concentrated in the unipotent fibre (3.24), we can define an associated torus embedding. Recall: The rational polyhedral cone decomposition is determined by its restriction to any $(U_{\mathbb{R}})(-1) \times \rho = C(\mathbb{D}^0, P_{\mathbf{X}}) \times \rho$ by the arithmeticity condition (3.20, 3.). Therefore this determines a torus embedding (see [FC, IV, §2] for the integral case):

$$M^K(\mathbf{X}/W_{\mathbf{X}}[U, 0]) \hookrightarrow M_{\Delta}^K(\mathbf{X}/W_{\mathbf{X}}[U, 0])$$

and a corresponding toroidal embedding of the torsor:

$$M^{K'}(\mathbf{X}) \hookrightarrow M_{\Delta}^{K'}(\mathbf{X})$$

(here K' and K are related by $K/U = K'/W$ and $K \cap U(\mathbb{A}^{(\infty)}) = K' \cap U(\mathbb{A}^{(\infty)})$.)

Main Theorem 6.9. *Let $p \neq 2$ be a prime. Assume Main Conjecture 6.6.*

There is a unique (up to unique isomorphism) extension of the pseudo-functor M to the category of p -ECMSD of Abelian type — with sufficiently fine smooth (w.r.t. the respective K) and projective Δ — to the category of Deligne-Mumford stacks over reflex rings, satisfying

1. *The morphism $M(\text{id}, 1) : M^K(\mathbf{X}) \rightarrow M_{\Delta}^K(\mathbf{X})$ is an open embedding. If $(\gamma, \rho) : \mathbf{Y} \rightarrow \mathbf{X}$ is an embedding,*

$$M(\gamma, \rho) \times \pi_{\mathbf{Y}} : M_{\Delta}^{K'}(\mathbf{Y}) \rightarrow M_{\Delta}^K(\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathcal{O}_{\mathbf{Y}})$$

is a normalization map followed by a closed embedding.

2. If $\Delta = \Delta^0$ (i.e. Δ is concentrated in the unipotent fibre, 3.24), $M(\Delta^K \mathbf{X})$ is the toroidal embedding described in 6.8.
3. $M(\Delta^K \mathbf{X})$ possesses a stratification into mixed Shimura varieties (stacks)

$$\coprod_{\substack{[(\iota, \rho) : \Delta'^{K'} \mathbf{B} \Rightarrow \Delta^K \mathbf{X}] \\ [\sigma] \in \Sigma(\iota, \rho)}} [\text{Stab}_\Gamma([\sigma]) \backslash M(\Delta^{K[\sigma]} \mathbf{B}_{[\sigma]})].$$

Here $[(\iota, \rho) : \Delta'^{K'} \mathbf{B} \Rightarrow \Delta^K \mathbf{X}]$ in the first line means equivalence classes of boundary components of $\Delta^K \mathbf{X}$ (3.23).

4. For each boundary map (3.23)

$$(\iota, \rho) : \Delta'^{K'} \mathbf{B} \Rightarrow \Delta^K \mathbf{X},$$

and $\sigma \in \Delta'$ such that $\sigma \subset C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho'$ for some $\mathbb{D}_{\mathbf{X}}^0$ and ρ' , there is a boundary isomorphism

$$M(\iota, \rho) : [\text{Stab}_\Gamma([\sigma]) \backslash \widehat{M(\Delta'^{K'} \mathbf{B})}] \rightarrow \widehat{M(\Delta^K \mathbf{X})}$$

of the formal completions along the boundary stratum

$$[\text{Stab}_\Gamma([\sigma]) \backslash M(\Delta^{K[\sigma]} \mathbf{B}_{[\sigma]})].$$

The complexification of $M(\iota, \rho)$ converges in a neighborhood of the boundary stratum and is in the interior, via the identification with the complex analytic mixed Shimura varieties given by a quotient of the map

$$\mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}} \times (P_{\mathbf{B}}(\mathbb{A}^{(\infty)})/K') \rightarrow \mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)$$

induced by the closed embedding $P_{\mathbf{B}} \hookrightarrow P_{\mathbf{X}}$, where we considered $\mathbb{D}_{\mathbf{B} \Rightarrow \mathbf{X}} \subseteq \mathbb{D}_{\mathbf{X}}$ as a subset of $\mathbb{D}_{\mathbf{B}}$ via the analytic boundary map (3.14).

The stratification is compatible with the functoriality, i.e. the morphisms $M(\cdot)$ induce morphisms of the strata of the same type again.

We have in addition:

5. $M(\Delta^K \mathbf{X})$ is smooth.
6. $M(\Delta^K \mathbf{X})$ is proper, if Δ is complete.
7. $M(\Delta^K \mathbf{X})$ is a projective scheme, if Δ is complete and K is neat.
8. The complement D of the open stratum $M(\Delta^K \mathbf{X})$ in $M(\Delta^K \mathbf{X})$ — the union of all lower dimensional strata in 3. — is a smooth divisor of normal crossings on $M(\Delta^K \mathbf{X})$.

Remark 6.10. *The stratification in 3. is indexed by pairs of an equivalence class of boundary components $[(\iota, \rho) : \Delta' \mathbf{B} \Rightarrow \Delta \mathbf{X}]$ and a class $[\sigma] \in \Sigma(\iota, \rho)$. The set of these pairs is just isomorphic to the set of double cosets $P_{\mathbf{X}}(\mathbb{Q}) \backslash \Delta / K$. The bijection is as follows. Each $\sigma \in \Delta$ is supported on a $C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \times \rho$. \mathbf{B} and ρ determine a boundary component $(\iota, \rho) : \Delta' \mathbf{B} \Rightarrow \Delta \mathbf{X}$ and the class of the image of σ under restriction to Δ' lies in $\Sigma(\iota, \rho)$. The stratum $M^{(K_{[\sigma]} \mathbf{B}_{[\sigma]})}$ is contained in the closure of the stratum $M^{(K_{[\tau]} \mathbf{B}_{[\tau]})}$, if and only if (up to a change of representatives in $P_{\mathbf{X}}(\mathbb{Q}) \backslash \Delta / K$) τ is a face of σ .*

6.11. There exist integral canonical models of the compact duals as well, which are defined as conjugacy classes of quasi-parabolic subgroup schemes. Let p be a prime and \mathbf{X} be p -integral mixed Shimura data. Let \mathcal{O} be the reflex ring of it at p and $S = \text{spec}(\mathcal{O})$. The closure \overline{M} of the conjugacy class of the morphisms $u_x, x \in \mathbb{D}_{\mathbf{X}}$ is defined over \mathcal{O} by Definition 6.2. It corresponds to a section $t' \in \mathcal{FTYPE}(S)$.

The image of \overline{M} via qpar is a fibre of the morphism ‘type’ above a section $t : S \rightarrow \mathcal{TYPE}$. We define $M^{\vee}(\mathbf{X})$ to be this fibre. It may, as a scheme over $\text{spec}(\mathbb{Z}_{(p)})$, be seen as an open and closed subscheme of \mathcal{QPAR} itself. We understand here the action of $P_{\mathbf{X}}$ on \mathcal{QPAR} by conjugation on the right (contrary to 2.8). The action fixes the morphism ‘type’, hence we have an induced operation on $M^{\vee}(\mathbf{X})$.

Main Theorem 6.12. *The construction above yields a functor from p -MSD to schemes over reflex rings (defined analogously). $M^{\vee}(\mathbf{X})$ is a smooth connected quasi-projective right $P_{\mathbf{X}, \mathcal{O}_{\mathbf{X}}}$ -homogeneous scheme over S called **the dual** of \mathbf{X} (depending only on $P_{\mathbf{X}}$ and $h(\mathbb{D}_{\mathbf{X}})$) with the following properties:*

1. *If \mathbf{X} is pure, $M^{\vee}(\mathbf{X})$ is projective.*
2. *Let L be a free $\mathbb{Z}_{(p)}$ -module of finite dimension, acted on faithfully by $P_{\mathbf{X}}$. $M^{\vee}(\mathbf{X})$ (considered as scheme over $\mathcal{O}_{\mathbf{X}}$) represents*

$$S' \mapsto \{ \text{filtrations } \{F^{\bullet}\} \text{ on } L_{S'} \text{ compatible with } (P_{\mathbf{X}, S'}, L_{S'}) \text{ of type } t' \},$$

compatible with $P_{\mathbf{X}, \mathcal{O}_{\mathbf{X}}}$ -action (defined on L as $v \cdot g := g^{-1}v$).

3. *For each map of p -MSD $\gamma : \mathbf{Y} \rightarrow \mathbf{X}$,*

$$M^{\vee}(\gamma) \times \pi_{\mathbf{Y}} : M^{\vee}(\mathbf{Y}) \rightarrow M^{\vee}(\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathcal{O}_{\mathbf{Y}})$$

is homogenous and is a closed embedding, if γ is an embedding.

4. *For each p -integral boundary map $\iota : \mathbf{B} \Rightarrow \mathbf{X}$, there is a $P_{\mathbf{B}}$ -equivariant open embedding*

$$M^{\vee}(\iota) : M^{\vee}(\mathbf{B}) \hookrightarrow M^{\vee}(\mathbf{X}).$$

5. *There is an $P_{\mathbf{X}}(\mathbb{R})W_{\mathbf{X}}(\mathbb{C})$ -equivariant open embedding (Borel embedding)*

$$h(\mathbb{D}_{\mathbf{X}}) \hookrightarrow (M^{\vee}(\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathbb{C}))^{an}.$$

It is an isomorphism, if $G_{\mathbf{X}}$ (the reductive part of $P_{\mathbf{X}}$) is a torus. These embeddings are compatible with the embeddings in 4., resp. (3.18, 1.)⁵. Furthermore they are compatible with morphisms of Shimura data.

Remark 6.13. For a boundary component $\mathbf{B} \Rightarrow \mathbf{X}$, there is also an embedding

$$M^{\vee}(\mathbf{B}/W_{\mathbf{B}}) \hookrightarrow M^{\vee}(\mathbf{X})$$

into (in general not onto) the complement of the image of $M^{\vee}(\mathbf{B})$. The image of the composition with the Borel embedding $h(\mathbb{D}_{\mathbf{B}}/W_{\mathbf{B}}) \hookrightarrow M^{\vee}(\mathbf{B}/W_{\mathbf{B}})(\mathbb{C})$ is the boundary component of $h(\mathbb{D}_{\mathbf{X}})$ associated with Q (the parabolic associated with \mathbf{B}) in the sense of [AMRT], if \mathbf{X} is pure.

The following Main Theorem 6.14 comprises the arithmetic theory of automorphic vector bundles on toroidal compactifications (see [18, Theorem 3.5.2] for a more concrete formulation of this). We will later derive a very concrete classical q -expansion principle (8.7). The theorem includes many kinds of classical results on CM-values of integral (at p of good reduction) modular forms on mixed Shimura varieties.

First, by taking quotients, we have a pseudo-functor $\mathbf{X} \rightarrow [M^{\vee}(\mathbf{X})/P_{\mathbf{X}, \mathcal{O}_{\mathbf{X}}}]$ from p -MSD to the 2-category of Artin stacks over reflex rings (defined analogously to above). It can be extended to p -ECMSD by forgetting K and Δ .

Main Theorem 6.14. Let $p \neq 2$ be a prime. Assume Main Conjecture 6.6. There is a unique (up to unique isomorphism) pseudo-natural transformation

$$\Xi_{\Delta}^K(\mathbf{X}) : M_{\Delta}^K(\mathbf{X}) \rightarrow [M^{\vee}(\mathbf{X})/P_{\mathbf{X}, \mathcal{O}_{\mathbf{X}}}] ,$$

and 2-isomorphism (or modification) of its restriction to p -EMSD:

$$(\Xi_{\Delta}^K(\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathbb{C}))^{an} \rightarrow \Xi_{\mathbb{C}}^K(\mathbf{X}),$$

where $\Xi_{\mathbb{C}}^K(\mathbf{X})$ is the complex analytic morphism described by the diagram:

$$\begin{array}{c} [P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K] \\ \uparrow \\ [P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K] \\ \downarrow \\ (M^{\vee}(\mathbf{X}) \times_{\text{spec}(\mathcal{O}_{\mathbf{X}})} \text{spec}(\mathbb{C}))^{an}, \end{array}$$

satisfying:

⁵going in different directions!

1. Let $[\iota, \rho] : {}^{K'}_{\Delta'} \mathbf{B} \Rightarrow {}^K_{\Delta} \mathbf{X}$ be a boundary map and $\sigma \in \Delta'$ such that $\sigma \in C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}})$, as in 6.7.
There is a 2-isomorphism $[\iota, \rho]_{\Xi}$ fitting into the diagram

$$\begin{array}{ccc}
\widehat{\text{Stab}_{\Gamma}([\sigma]) \backslash M({}^{K'}_{\Delta'} \mathbf{B})} & \xrightarrow{\Xi({}^{K'}_{\Delta'} \mathbf{B})} & [M^{\vee}(\mathbf{B})/P_{\mathbf{B}}] \\
\downarrow M(\iota, \rho) & \searrow [\iota, \rho]_{\Xi} & \downarrow M^{\vee}(\iota) \\
\widehat{M({}^K_{\Delta} \mathbf{X})} & \xrightarrow{\Xi({}^K_{\Delta} \mathbf{X})} & [M^{\vee}(\mathbf{X})/P_{\mathbf{X}}]
\end{array}$$

where the formal completion is taken along

$$\left[\text{Stab}_{\Gamma}([\sigma]) \backslash M({}^{K'[\sigma]}_{\Delta'[\sigma]} \mathbf{B}_{[\sigma]}) \right],$$

such that $[\iota, \rho]_{\Xi, \mathbb{C}}$ converges in the interior and is the obvious 2-morphism over \mathbb{C} .

2. (canonicity) For $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ (and $\Delta = 0$), $\Xi({}^K \mathbf{X})$ is described by the diagram:

$$[{}^K \mathbf{X}\text{-}\mathbf{L}\text{-}\mathbf{mot}] \leftarrow [{}^K \mathbf{X}\text{-}\mathbf{L}\text{-}\mathbf{mot}\text{-}\mathbf{triv}] \rightarrow M^{\vee}(\mathbf{X}),$$

(see 7.3) given by the forgetful left arrow (which is a right $P_{\mathbf{X}}$ -torsor) and the right arrow given by transport of Hodge filtration. The identification with $\Xi({}^K \mathbf{X})$ is the one given by the period construction (7.5).

3. For Δ concentrated in the unipotent fibre (3.24), Ξ is given as follows: Recall from 6.8 that $M({}^{K'} \mathbf{X}) \rightarrow M({}^{K'} \mathbf{X}/U)$ is a torsor under the torus $T = M({}^K \mathbf{X}/W_{\mathbf{X}}[U, 0])$ over $M({}^K \mathbf{X}/W_{\mathbf{X}})$ for K and K' chosen as in 6.8. $\Xi({}^{K'} \mathbf{X})$ may etale locally (say on $S \rightarrow M({}^{K'/U} \mathbf{X}/U_{\mathbf{X}})$) be trivialized T -invariantly. $\Xi({}^{K'}_{\Delta} \mathbf{X})$ is then the unique extension of $\Xi({}^{K'} \mathbf{X})$ to the torus embedding $M({}^{K'}_{\Delta} \mathbf{X})$ by means of extending any such T -invariant trivialization.

7. Mixed Shimura varieties of symplectic type

7.1. If ${}^K \mathbf{X}$ is p -EMSD and L is a representation of $P_{\mathbf{X}}$, there should exist a category fibred in groupoids (over $\text{spec}(\mathcal{O}_{\mathbf{X}})$)

$$[{}^K \mathbf{X}\text{-}\mathbf{L}\text{-}\mathbf{mot}]$$

of motives with “ $(P_{\mathbf{X}}, L)$ -structure” (see e.g. the introduction to [18]). It should yield (up to taking several connected components) a canonical integral model of the Shimura variety described by 6.5. This can be made precise for P.E.L.-type

Shimura data and the associated standard representation. We will need only the case of the symplectic Shimura data and certain representations (sometimes different ones, however) of their groups.

For each of the symplectic mixed Shimura data (4.4) we will explicitly describe a moduli problem in terms of 1-motives which belongs to the standard representation of the underlying group scheme. These moduli problems are representable by a Deligne-Mumford stack and give an explicit description of the canonical integral models. In the case of the morphism Ξ encoding the theory of automorphic vector bundles, the corresponding explicit model was already used to normalize it for all mixed Shimura varieties of Abelian type (cf. Main Theorem 6.14).

For the language of 1-motives, as used here, we refer the reader to [18, Chapter 5]. We will denote a motive M by $[\underline{Y} \xrightarrow{\mu} G]$ or by the symmetric description $(A, A^\vee, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu)$ interchangingly.

Lemma 7.2. *The reflex field of the symplectic mixed Shimura data (4.4) is \mathbb{Q} .*

Definition 7.3. *We define the following stacks over $\mathbb{Z}_{(p)}$:*

1. Recall the mixed Shimura datum $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ from 4.4. Let $L = I^* \oplus L_0 \oplus I$ with standard representation (cf. 4.3) of $P_{\mathbf{X}}$ on it, where $L_{0, \mathbb{Z}_{(p)}}$ is a lattice of dimension $2g_0$ with unimodular symplectic form (if $g_0 \neq 0$), $I_{\mathbb{Z}_{(p)}}$ is any lattice, and $K \subset \mathrm{PSp}(L_{\mathbb{A}^{(\infty)}})$ is an admissible compact open subgroup. Assume that $L \neq 0$. We define

$$[{}^K \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I] \text{-} L \text{-} \mathbf{mot}]$$

as the stack of the following data:

- (a) a $\mathbb{Z}_{(p)}$ -scheme S ,
- (b) a 1-motive $M = (A, A^\vee, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu)$ over S , with $\dim(A) = g_0$,
- (c) a p -polarization of some degree $d \in \mathbb{Z}_{(p)}^*$

$$\psi_1 : A \rightarrow A^\vee$$

and an isomorphism

$$\psi_2 : \underline{X} \otimes \mathbb{Z}_{(p)} \rightarrow \underline{Y} \otimes \mathbb{Z}_{(p)}$$

of the same degree d , such that they give a p -polarization of M ,

- (d) an isomorphism $\rho : (I_{\mathbb{Z}_{(p)}}^*)_S \cong \underline{Y} \otimes \mathbb{Z}_{(p)}$,
- (e) a $K^{(p)}$ -level structure (as above)

$$\xi \in \underline{\mathrm{Iso}}_{(\mathrm{USp}(L_0, I), L)}(L_{\mathbb{A}^{(\infty, p)}}, H^{\mathrm{et}}(M, \mathbb{A}^{(\infty, p)}))/K^{(p)},$$

where the isomorphisms in the etale sheaf on the right have to be compatible with the $(\mathrm{USp}(L_{0, \mathbb{Z}_{(p)}}, I_{\mathbb{Z}_{(p)}}), L_{\mathbb{Z}_{(p)}})$ -structure (see below) on both parameters.

Here the $(\mathrm{USp}(L_{0,\mathbb{A}^{(\infty,p)}}, I_{\mathbb{A}^{(\infty,p)}}), L_{\mathbb{A}^{(\infty,p)}})$ -structure is given as follows: The p -polarization induces an isomorphism

$$H^{et}(M, \mathbb{A}^{(\infty,p)}) \rightarrow H^{et}(M, \mathbb{A}^{(\infty,p)})^*(1).$$

Choosing some isomorphism $\mathbb{Z}_{(p)}^{et}(1) \cong \mathbb{A}^{(\infty,p)}$, we get an alternating form on $H^{et}(M, \mathbb{A}^{(\infty,p)})$ up to a scalar. The weight filtration satisfies W_{-2} totally isotropic and $W_{-1} = (W_{-2})^\perp$. Furthermore, we have (via ρ) an isomorphism

$$\mathrm{gr}^0(H^{et}(M, \mathbb{A}^{(\infty,p)})) = \underline{Y} \otimes_{\mathbb{Z}} \mathbb{A}^{(\infty,p)} = I_{\mathbb{A}^{(\infty,p)}}^*$$

(i.e. a USp -structure, see 4.8).

Morphisms are $\mathbb{Z}_{(p)}$ -morphisms of schemes $S \rightarrow S'$, together with an $\mathbb{Z}_{(p)}$ -isomorphism of the pullback of M with the pullback of M' to S , compatible with polarization (up to scalar), ρ 's and level structures.

2. With the same notation as in 1. we define

$$[{}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I] \text{-} L\text{-}\mathbf{mot}\text{-}\mathbf{triv}]$$

as the stack consisting of the same data (S, M, \dots) as before, but in addition with a trivialization

$$\gamma : L \otimes \mathcal{O}_S \rightarrow H^{dR}(M)$$

of the de Rham realization of M compatible with the $\mathrm{USp}(L_0, I)$ -structure on both sides. $P_{\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]} = \mathrm{USp}(L_0, I)$ acts on this stack on the right by translation of β .

This renders the stack $[{}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I] \text{-} L\text{-}\mathbf{mot}\text{-}\mathbf{triv}]$ into a right $\mathrm{USp}(L_0, I)$ -torsor over $[{}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I] \text{-} L\text{-}\mathbf{mot}]$.

3. Consider the Shimura datum $\mathbf{X} = \mathbf{H}_0[U, 0]$, a simple unipotent extension of \mathbf{H}_0 , where U is any lattice of dimension k . We have $P_{\mathbf{X}} = \mathbb{W}(U) \rtimes \mathbb{G}_m$. $P_{\mathbf{X}}$ acts on $L' = \mathbb{Z}_{(p)} \oplus U^*$ as in 4.9 (If $U = (I \otimes I)^s$ we get one of the basic symplectic Shimura data but L' is not the standard representation). Let $K \subset \mathrm{PSp}(L_{\mathbb{A}^{(\infty)}})$ be an admissible compact open subgroup. We define

$$[{}^K\mathbf{H}_0[U, 0] \text{-} L'\text{-}\mathbf{mot}]$$

as the stack of the following data:

- (a) a $\mathbb{Z}_{(p)}$ -scheme S ,
- (b) a 1-motive of the form $M = [\underline{Y} \longrightarrow T]$ (without part of weight -1),
- (c) an isomorphism $\rho : (U_{\mathbb{Z}_{(p)}}^*)_S \cong \underline{Y} \otimes \mathbb{Z}_{(p)}$,
- (d) an isomorphism $\rho' : (\mathbb{Z}_{(p)})_S \rightarrow \underline{X} \otimes \mathbb{Z}_{(p)}$,
- (e) a $K^{(p)}$ -level structure (as above),

$$\xi \in \underline{\mathrm{Iso}}_{(P_{\mathbf{X}}, L_{\mathbb{Z}_{(p)}})}(L'_{\mathbb{A}^{(\infty,p)}}, H^{et}(M, \mathbb{A}^{(\infty,p)}))/K^{(p)},$$

where the isomorphisms in the etale sheaf on the right have to be compatible with the obvious $(P_{\mathbf{X}}, L'_{\mathbb{Z}_{(p)}})$ -structure on both parameters.

Isomorphisms are morphisms over $\text{spec}(\mathbb{Z}_{(p)})$ of schemes $S \rightarrow S'$ together with an isomorphism of M with the pullback of M' compatible with ρ 's and level structures.

4. With the same notation as in 3. we define

$$[{}^K \mathbf{H}_0[U, 0]\text{-}L'\text{-}\mathbf{mot}\text{-}\mathbf{triv}]$$

consisting of the same data (S, M, \dots) as before, but in addition with a trivialization

$$\gamma : L' \otimes \mathcal{O}_S \rightarrow H^{dR}(M)$$

of the de Rham realization of M compatible with the $(P_{\mathbf{H}_0[U, 0]}, L')$ -structure on both sides.

$P_{\mathbf{H}_0[U, 0]}$ acts on this stack on the right by translation of β .

This renders the stack $[{}^K \mathbf{H}_0[U, 0]\text{-}L'\text{-}\mathbf{mot}\text{-}\mathbf{triv}]$ into a right $P_{\mathbf{X}}$ -torsor over $[{}^K \mathbf{H}_0[U, 0]\text{-}L'\text{-}\mathbf{mot}]$.

Actually there is a similar description of stacks

$$[{}^K \mathbf{X}\text{-}L\text{-}\mathbf{mot}\text{-}\mathbf{triv}] \quad [{}^K \mathbf{X}\text{-}L\text{-}\mathbf{mot}]$$

where \mathbf{X} is any of the symplectic Shimura data.

Definition 7.4. Let \mathbf{X} be one of the symplectic mixed Shimura data of Definition 7.3 with natural representation of $P_{\mathbf{X}}$ on $L_{\mathbb{Z}_{(p)}}$. Furthermore for each $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty, p)})$ and admissible compact open subgroups K_1, K_2 , such that $K_1^\rho \subseteq K_2$ we have a map

$$[{}^{K_1} \mathbf{X}\text{-}L\text{-}\mathbf{mot}] \rightarrow [{}^{K_2} \mathbf{X}\text{-}L\text{-}\mathbf{mot}]$$

by multiplication of the level-structure by ρ from the right.

7.5. For $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ with $g_0 \neq 0$, $[{}^K \mathbf{X}\text{-}L\text{-}\mathbf{mot}]$ yields a model of the analytic mixed Shimura variety:

$$[P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) / K].$$

The identification is given as follows. Let an object

$$(M, \dots) \in [\text{spec}(\mathbb{C})\text{-}{}^K \mathbf{X}\text{-}L\text{-}\mathbf{mot}]$$

be given. Choose an isomorphism $\beta : H^B(M, \mathbb{Q}) \rightarrow L_{\mathbb{Q}}$ (compatible with $(P_{\mathbf{X}}(\mathbb{Q}), L_{\mathbb{Q}})$ -structures on both sides). Consider the mixed Hodge structure F^\bullet, W_\bullet on $H^B(M, \mathbb{C})$, where F^\bullet comes via the comparison isomorphism

$$\rho_{dR, B} : H^{dR}(M, \mathbb{C}) \rightarrow H^B(M, \mathbb{C})$$

from the Hodge filtration of H^{dR} . The Hodge structure $\beta(F^\bullet), \beta(W_\bullet)$ is given by a morphism $h : \mathbb{S}_{\mathbb{C}} \rightarrow \text{GL}(L_{\mathbb{C}})$ which factors through the used representation of $P_{\mathbf{X}, \mathbb{C}}$ on $L_{\mathbb{C}}$ because the isomorphism was chosen to be compatible with the

$P_{\mathbf{X}}$ -structures. The level structure ξ gives an element $\beta_{\rho_{et},B}\xi \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$. The class of $(h, \beta_{\rho_{et},B}\xi)$ is well defined.

The identification of a moduli point of the triv-category fibre over $\text{spec}(\mathbb{C})$ with

$$P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K$$

sends a pont (M, \dots, γ) , where (M, \dots) is as before and $\gamma : L_{\mathbb{C}} \rightarrow H_1^{dR}(M)$ is a trivialization to the (well-defined) point $(h, \beta_{\rho_{dR},B}\gamma, \beta_{\rho_{et},B}\xi)$.

If $g_0 = 0$, but $I \neq 0$, i.e. in the case of the mixed Shimura datum $\mathbf{X} = \mathbf{H}_0[(I \otimes I)^s, 0]$, we have two descriptions as a moduli problem, according to the representations $L = I \oplus I^*$ or $L' = \mathbb{Z}_{(p)} \oplus (I \otimes I)^s$.

In the first case (representation L) $\mathbb{D}_{\mathbf{X}}$ is given as the product of the set of Hodge structures as above and $\mathbb{D}_{\mathbf{H}_0} = \text{Hom}(\mathbb{Z}, \mathbb{Z}(1))$. We get the point $(\alpha, h, \beta_{\rho_{dR},B}\gamma, \beta_{\rho_{et},B}\xi)$ where α is determined as follows: The chosen isomorphism $\beta : H_1(M, \mathbb{Q}) \cong L_{\mathbb{Q}}$, pins down a symplectic form on $H_1(M, \mathbb{Q})$. The comparison with the natural $\mathbb{Q}(1)$ -valued form on $H_1(M, \mathbb{Q})$ (which coincides up to scalar) induces an isomorphism $\mathbb{Q} \rightarrow \mathbb{Q}(1)$ of the form $q\alpha$ with $q \in \mathbb{Q}_{>0}$. In the second case (representation L') — and this even includes $I = 0$ — the isomorphism $W_{-1}(\beta)^{-1} : \mathbb{Q} \cong H_1(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}(1)$ is of the form $q\alpha$, as before. This determines in both cases an $\alpha \in \text{Hom}(\mathbb{Z}, \mathbb{Z}(1))$.

The following theorem is due to Riemann, Mumford, Artin, Deligne, etc. in the pure case, due to Pink in the mixed case (rational case) and was extended to the integral mixed case in the thesis of the author [18]:

Theorem 7.6. *For each of the p -integral mixed Shimura data of symplectic type \mathbf{X} and each admissible compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$*

$$[{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mot}],$$

considered as a model of the analytic mixed Shimura variety via the construction 7.5, is representable by the canonical model $M({}^K\mathbf{X})$ (compatible with the $P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$ -action) which is a smooth, Deligne-Mumford stack of finite type over $\text{spec}(\mathbb{Z}_{(p)})$. It is a quasi-projective scheme, if K is neat.

The stack

$$[{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mot}\text{-}\mathbf{triv}]$$

is representable by a right $P_{\mathbf{X}}$ -torsor over $[{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mot}]$.

7.7. Note that the torsor $[{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mot}\text{-}\mathbf{triv}]$ was used to “normalize” the theory of the of the morphism Ξ in 6.14. Involved in the construction of Ξ there is a $P_{\mathbf{X}}$ -equivariant morphism $P({}^K\mathbf{X}) \rightarrow M^{\vee}(\mathbf{X})$ which in the modular description is given by transport of the Hodge filtration to $L_{\mathcal{O}_S}$ by means of the trivialization β (and then taking its fixing quasi-parabolic, cf. 6.12, 2.).

We have the following ‘integral’ variants of the foregoing stacks:

Definition 7.8. *Fixing an $N \in \mathbb{N}$, we define the following stacks over $\text{spec}(\mathbb{Z}[1/N])$.*

1. Let $L_{0,\mathbb{Z}}$ be a with unimodular symplectic form of dimension $2g_0$. Let $I_{\mathbb{Z}}$ be another lattice and $L_{\mathbb{Z}} = I_{\mathbb{Z}}^* \oplus L_{0,\mathbb{Z}} \oplus I_{\mathbb{Z}}$ as above. Assume again $L \neq 0$. We define

$$[{}^N\mathbf{H}_{g_0}[(I_{\mathbb{Z}} \otimes I_{\mathbb{Z}})^s, L_{0,\mathbb{Z}} \otimes I_{\mathbb{Z}}] - L_{\mathbb{Z}}\text{-}\mathbf{mot}]$$

as the stack of the following data:

- (a) a scheme S over $\mathrm{spec}(\mathbb{Z}_{(p)})$,
- (b) a 1-motive $M = (A, A^{\vee}, \underline{X}, \underline{Y}, \alpha, \alpha^{\vee}, \nu)$ over S , where $\dim(A) = g$,
- (c) a principal polarization

$$\psi_1 : A \rightarrow A^{\vee}$$

and an isomorphism

$$\psi_2 : \underline{X} \rightarrow \underline{Y}$$

such that they induce a polarization of M ,

- (d) an isomorphism $\rho : (I_{\mathbb{Z}}^*)_S \cong \underline{Y}$,
- (e) a level- N -structure

$$\xi \in \mathrm{Iso}(\mathrm{USp}(L_{0,\mathbb{Z}/N\mathbb{Z}}, I_{\mathbb{Z}/N\mathbb{Z}}), L_{\mathbb{Z}/N\mathbb{Z}})(L_{\mathbb{Z}/N\mathbb{Z}}, H^{et}(M, \mathbb{Z}/N\mathbb{Z})).$$

This means that the isomorphisms have to be compatible with the $(\mathrm{USp}(L_{0,\mathbb{Z}/N\mathbb{Z}}, I_{\mathbb{Z}/N\mathbb{Z}}), L_{\mathbb{Z}/N\mathbb{Z}})$ -structure (see the adelic case) on both parameters.

Morphisms are as in the adelic case with the difference, that the $\mathbb{Z}_{(p)}$ -morphism of the 1-motives has to be an isomorphism. (Note: In particular ψ_Y is determined by the ρ 's and ψ_X by them and the polarization.)

2. We define the stack

$$[{}^N\mathbf{H}_{g_0}[(I_{\mathbb{Z}} \otimes I_{\mathbb{Z}})^s, L_{0,\mathbb{Z}} \otimes I_{\mathbb{Z}}] - L_{\mathbb{Z}}\text{-}\mathbf{mot-triv}]$$

as before with a trivialization of de Rham.

3. Let $L'_{\mathbb{Z}} = \mathbb{Z} \oplus U_{\mathbb{Z}}^*$ be as in 7.3, 3. We define the stack

$$[{}^N\mathbf{H}_0[U_{\mathbb{Z}}, 0] - L'_{\mathbb{Z}}\text{-}\mathbf{mot}]$$

as the category of the following data:

- (a) a scheme S over $\mathrm{spec}(\mathbb{Z}_{(p)})$,
- (b) a 1-motive of the form $M = [\underline{Y} \longrightarrow T]$,
- (c) an isomorphism $\rho : (U_{\mathbb{Z}}^*)_S \cong \underline{Y}$,
- (d) an isomorphism $\rho' : (\mathbb{Z})_S \cong \underline{X}$,
- (e) a level- N -structure

$$\xi \in \underline{\mathrm{Iso}}_{(P_{\mathbf{X}}, L'_{\mathbb{Z}/N\mathbb{Z}})}(L'_{\mathbb{Z}/N\mathbb{Z}}, H^{et}(M, \mathbb{Z}/N\mathbb{Z})),$$

where the isomorphisms in the etale sheaf on the right have to be compatible with the obvious $(P_{\mathbf{X}}, L'_{\mathbb{Z}/N\mathbb{Z}})$ -structure on both parameters. Morphisms are as in the adelic case, with the difference that the $\mathbb{Z}_{(p)}$ -morphism of the 1-motives has to be an isomorphism.

4. We define the stack

$$[{}^N\mathbf{H}_0[U_{\mathbb{Z}}, 0] \text{-} L'_{\mathbb{Z}} \text{-} \mathbf{mot-triv}]$$

as before with a trivialization of de Rham.

5. We define the stack

$$[{}^{N,0}\mathbf{H}_0[U_{\mathbb{Z}}, 0] \text{-} L'_{\mathbb{Z}} \text{-} \mathbf{mot}]$$

where $L'_{\mathbb{Z}} = \mathbb{Z} \oplus U_{\mathbb{Z}}^*$ as the category of the following data (S, M, \dots, ξ) as before, but with

$$\xi \in \underline{\mathrm{Iso}}(\mathbb{Z}/N\mathbb{Z}, H^{\mathrm{et}}(\mathrm{gr}_{-2} M, \mathbb{Z}/N\mathbb{Z})),$$

only.

6. We define the stack

$$[{}^{N,0}\mathbf{H}_0[U_{\mathbb{Z}}, 0] \text{-} L'_{\mathbb{Z}} \text{-} \mathbf{mot-triv}]$$

as before with a trivialization of de Rham.

Remark 7.9. For $U = (I \otimes I)^s \neq 0$, we explicitly describe the “change-of-representation equivalences”:

$$[{}^K\mathbf{H}_0[(I \otimes I)^s, 0] \text{-} L = I^* \oplus I \text{-} \mathbf{mot}] \rightarrow [{}^K\mathbf{H}_0[U, 0] \text{-} L' = \mathbb{Z}_{(p)} \oplus U^* \text{-} \mathbf{mot}].$$

and

$$[{}^K\mathbf{H}_0[(I \otimes I)^s, 0] \text{-} L = I^* \oplus I \text{-} \mathbf{mot-triv}] \rightarrow [{}^K\mathbf{H}_0[U, 0] \text{-} L' = \mathbb{Z}_{(p)} \oplus U^* \text{-} \mathbf{mot-triv}].$$

as follows: Let a $(M = (0, 0, \underline{X}, \underline{Y}, 0, 0, \nu), 0, \psi_2, \rho, \xi)$ over S be given. ν can via polarization and ρ be considered as a morphism $\mathrm{Sym}^2(I^*)_S \rightarrow \mathbb{G}_{m,S}$. This gives rise to a motive M'' and a ρ required for the second category (ρ' is given by the canonical identification $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$). The level structures translate as follows. It is easy to check that the realizations $H^{\mathrm{et}}(M'')$, $H^{\mathrm{dR}}(M'')$ and $H^B(M'')$ (when defined), together with their $(\mathrm{USp}(0, I), \mathbb{Z}_{(p)} \oplus \mathrm{Sym}^2(I^*))$ -structures, are obtained from the corresponding realizations of M with their $(\mathrm{USp}(0, I), I \oplus I^*)$ -structures by the procedure described in 4.9. Since the procedure described in 4.9 applied to $I \oplus I^*$ gives $\mathbb{Z}_{(p)} \oplus \mathrm{Sym}^2(I^*)$ canonically, we may translate the K -level structure or trivialization of H^{dR} accordingly.

Proposition 7.10. Let S be a scheme over $\mathrm{spec}(\mathbb{Z}_{(p)})$. For each of the p -integral mixed symplectic Shimura data as above, we have equivalences

$$[{}^N\mathbf{X} \text{-} L_{\mathbb{Z}} \text{-} \mathbf{mot}] \rightarrow [{}^{K(N)}\mathbf{X} \text{-} L \text{-} \mathbf{mot}],$$

where

$$K(N) = \{g \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) \mid gL_{\widehat{\mathbb{Z}}} = L_{\widehat{\mathbb{Z}}}, g \equiv \mathrm{id} \ (N)\}.$$

We have also a similar equivalences for $\mathbf{X} = \mathbf{H}_0[U, 0]$ and the representation L' , as well as

$$[{}^{N,0}\mathbf{X} \text{-} L'_{\mathbb{Z}} \text{-} \mathbf{mot}] \rightarrow [{}^{U_{\widehat{\mathbb{Z}}} \rtimes K(\mathbb{G}_m, N)}\mathbf{X} \text{-} L' \text{-} \mathbf{mot}].$$

Remark 7.11. *Since the $K(N)$ form a cofinal system of compact open groups, the integral categories give a complete description of the canonical models. In the case $\mathbf{X} = \mathbf{H}_0[U, 0]$, each $K(N)$ is conjugated to $U_{\widehat{\mathbb{Z}}} \rtimes K(\mathbb{G}_m, N)$, therefore already the integral categories with $(N, 0)$ -level structure suffice to describe the canonical models.*

Proof of the Proposition. We will prove this for the case $\mathbf{H}_{g_0}[(I_{\mathbb{Z}} \otimes I_{\mathbb{Z}})^s, L_{0, \mathbb{Z}} \otimes I_{\mathbb{Z}}]$, where $I_{\mathbb{Z}} \neq 0$, $L_{0, \mathbb{Z}} \neq 0$ — the other cases are degenerate special cases of this construction. We may assume that S is connected.

We first describe the functor. Let $[M, \psi_1, \psi_2, \rho, \xi]$ be an object of $[{}^N\mathbf{X}\text{-}L\text{-}\mathbf{mot}]$ over S . Choose a geometric point \bar{s} . ξ can be considered as an isomorphism

$$\xi : L_{\mathbb{Z}/N\mathbb{Z}} \rightarrow H^{et}(M_{\bar{s}}, \mathbb{Z}/N\mathbb{Z})$$

invariant under the action of $\pi_1^{et}(S, \bar{s})$ ($H^{et}(M, \mathbb{Z}/N\mathbb{Z})$ has to be constant). Choose some isomorphism

$$\delta : H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) \rightarrow L_{\widehat{\mathbb{Z}}^{(p)}}$$

compatible with USp -structures. Composing the reduction mod N with ξ , we get an element of $\mathrm{USp}(L_{0, \mathbb{Z}/N\mathbb{Z}}, I_{\mathbb{Z}/N\mathbb{Z}})$. Since USp is a smooth group scheme over \mathbb{Z} , by Hensel's Lemma we get a lift to $\mathrm{USp}(L_{\widehat{\mathbb{Z}}^{(p)}})$. Taking composition again with the chosen isomorphism we get a

$$\xi' : L_{\widehat{\mathbb{Z}}^{(p)}} \rightarrow H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}),$$

reducing mod N to ξ . It is well-defined mod $K(N)$ and the class is, by construction, invariant under $\pi_1^{et}(S, \bar{s})$.

This functor is faithful. It is full because an p -morphism of 1-motives which induces an isomorphism $H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) \cong H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)})$ must be a morphism. This follows from [18, Theorem 5.1.7].

Let on the other hand $[M, \psi_1, \psi_2, \rho, \xi']$ be an object of $[{}^{K(N)}\mathbf{X}\text{-}L\text{-}\mathbf{mot}]$ over S . Choose a geometric point \bar{s} .

If ξ' is represented by an isomorphism

$$L_{\widehat{\mathbb{Z}}^{(p)}} \rightarrow H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)})$$

then the object is in the image of the functor because of the following:

1. ξ' can be given as a lift of a ξ as above.
2. There is a principal polarization in the class of ψ' . For, there is a $d \in \mathbb{A}^{(\infty, p)}$, such that Ψ induces an isomorphism

$$H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) \rightarrow dH^{et}(M_{\bar{s}}^{\vee}, \widehat{\mathbb{Z}}^{(p)})$$

because ξ' is a morphism of USp -structures, hence a symplectic similitude. So we get a principal polarization by changing ψ' by $+d$ or $-d$, which must lie in $\mathbb{Z}_{(p)}^*$. Only one sign leads to a *polarization*.

3. ρ maps $I_{\mathbb{Z}}^*$ to \underline{Y} .

If ξ' is not represented by an isomorphism as above, we have to show that there exists an isogenous object with this property.

Composing with a p -isogeny $\in \mathbb{Z}_{(p)} \setminus \{0\}$, we may assume that there is an $c, p \nmid c$, and a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) & \xrightarrow{\xi^{-1}} & L_{\widehat{\mathbb{Z}}^{(p)}} & \longrightarrow & K_{\bar{s}} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) & \xrightarrow{[c]} & H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) & \longrightarrow & \ker([c]) \longrightarrow 0 \end{array}$$

Furthermore the operation of $\pi_1^{et}(S, \bar{s})$ induces one on $K_{\bar{s}}$, hence there is a finite etale group scheme $K \subset \ker([c])$ with fibre $K_{\bar{s}}$, and we have [18, Theorem 5.1.8] an isogeny $\psi : M \rightarrow M'$ with $\ker(\psi) = K$, hence a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) & \xrightarrow{\xi^{-1}} & L_{\widehat{\mathbb{Z}}^{(p)}} & \longrightarrow & K_{\bar{s}} \longrightarrow 0 \\ & & \parallel & & \uparrow (\xi')^{-1} & & \parallel \\ 0 & \longrightarrow & H^{et}(M_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) & \xrightarrow{\psi} & H^{et}(M'_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)}) & \longrightarrow & K_{\bar{s}} \longrightarrow 0 \end{array}$$

There is an ρ' because ξ is a morphism of USp -structures and hence Ψ_Y has to be an isomorphism of \mathbb{Z} -lattices! Therefore we get an isogenous object with the property that ξ' is an isomorphism $L_{\widehat{\mathbb{Z}}^{(p)}} \rightarrow H^{et}(M'_{\bar{s}}, \widehat{\mathbb{Z}}^{(p)})$. \square

7.12. Let U be any $\mathbb{Z}_{(p)}$ -lattice. Consider the mixed Shimura datum $\mathbf{X} = \mathbf{H}_0[U, 0]$. It plays the mayor role in the description of point-like boundary components of many pure Shimura varieties, in particular of those symplectic and also orthogonal type (cf. 4.5 and 5.5).

Consider the category (cf. Definition 7.3, 3.):

$$[{}^K \mathbf{X}\text{-}U^* \oplus \mathbb{Z}_{(p)}\text{-}\mathbf{mot}],$$

where K is any admissible compact open subgroup of $P_{\mathbf{H}_0[U, 0]}(\mathbb{A}^{(\infty)})$.

Recall that, by definition,

$$P_{\mathbf{H}_0[U, 0]} = \mathbb{W}(U) \rtimes \mathbb{G}_m \tag{5}$$

and that K contains the subgroup $U_{\widehat{\mathbb{Z}}} \rtimes K(M)$ for some $M \in \mathbb{Z}$ and some (saturated at p) lattice $U_{\mathbb{Z}} \subset U$. The splitting (5) determines and is determined by a $\mathbb{Z}_{(p)}$ -valued point v of the dual $M^\vee(\mathbf{X})$ (The preimage of \mathbb{G}_m under the splitting is the quasi-parabolic corresponding to the point).

The category

$$[{}^{U_{\widehat{\mathbb{Z}}} \rtimes K(M)} \mathbf{H}_0[U, 0]\text{-}U^* \oplus \mathbb{Z}_{(p)}\text{-}\mathbf{mot}]$$

is isomorphic to the integral category

$$[{}^{(M,0)}\mathbf{H}_0[U,0]-U_{\mathbb{Z}}^* \oplus \mathbb{Z}\text{-}\mathbf{mot}],$$

see 7.10. An object in the latter is isomorphic to (M, ξ) , where M is of the form

$$[\mu : U_{\mathbb{Z}}^* \rightarrow \mathbb{G}_m],$$

with a level structure $\xi : \mathbb{Z}/M\mathbb{Z} \rightarrow H_1^{et}(\mathbb{G}_m, \mathbb{Z}/M\mathbb{Z}) \cong \mathbb{G}_m[M]$ only.

We may identify

$$\mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \rightarrow \mathbb{D}_{\mathbf{H}_0[U,0]}$$

such that an element (α, u) is mapped to $(\alpha, u \circ h_0 \circ (-u))$, where $u \circ h_0 \circ (-u)$ is the u -conjugate (from the left) of the morphism $h_0 : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbf{H}_0[U,0]} = \mathbb{W}(U) \rtimes \mathbb{G}_m$ mapping z to $z\bar{z}$ in the second factor. The associated mixed Hodge structure is the u -translate of the one given by $F^0 = U_{\mathbb{C}}$.

The following Proposition — which is quite tautological — describes the associated canonical models. It will be used in the proof of the q -expansion principle (Theorem 8.7).

Proposition 7.13. *The S -valued points of the dual $M^{\vee}(\mathbf{X})(S)$ are in natural bijection with splittings $P_{\mathbf{X},S} \cong \mathbb{W}(U_S) \rtimes \mathbb{G}_{m,S}$.*

We have a commutative diagram (where all morphisms are functorial in S)

$$\begin{array}{ccc} [S^{-(M,0)}\mathbf{X}\text{-}\mathbb{Z} \oplus U_{\mathbb{Z}}^*\text{-}\mathbf{mot}\text{-}\mathbf{triv}] & \xrightarrow[\sim]{i_3} & (P \times_{\mathbb{Z}_{(p)}} \mathbb{G}_{m,\mathbb{Z}_{(p)}(\zeta_M)} \otimes U_{\mathbb{Z}})(S) \\ \downarrow p \quad \uparrow \text{can.} & & \downarrow p \quad \uparrow \text{can.} \\ [S^{-(M,0)}\mathbf{X}\text{-}\mathbb{Z} \oplus U_{\mathbb{Z}}^*\text{-}\mathbf{mot}] & \xrightarrow[\sim]{i_2} & (\mathbb{G}_{m,\mathbb{Z}_{(p)}(\zeta_M)} \otimes U_{\mathbb{Z}})(S) \\ \downarrow p & & \downarrow p \\ [S^{-M}\mathbf{H}_0\text{-}\mathbb{Z}\text{-}\mathbf{mot}] & \xrightarrow[\sim]{i_1} & \text{spec}(\mathbb{Z}_{(p)}(\zeta_M))(S) \end{array} \quad (6)$$

where ζ_M an (abstract) primitive M -th root of unity. Here the middle horizontal morphisms give morphisms of tori, and the topmost horizontal morphisms give morphisms of right P_S -torsors.

Over \mathbb{C} , the identifications 7.5 render this into the following commutative diagram

$$\begin{array}{ccc} P(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P(\mathbb{C}) \times P(\mathbb{A}^{(\infty)})/K & \xrightarrow[\sim]{j_3} & (P \times_{\mathbb{Z}_{(p)}} \mathbb{G}_{m,\mathbb{Z}_{(p)}(\zeta_M)} \otimes U_{\mathbb{Z}})(\mathbb{C}) \\ \downarrow p \quad \uparrow \text{can.} & & \downarrow p \quad \uparrow \text{can.} \\ P(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P(\mathbb{A}^{(\infty)})/K & \xrightarrow[\sim]{j_2} & (\mathbb{G}_{m,\mathbb{Z}_{(p)}(\zeta_M)} \otimes U_{\mathbb{Z}})(\mathbb{C}) \\ \downarrow p & & \downarrow p \\ \mathbb{Q}^* \backslash \mathbb{D}_{\mathbf{H}_0} \times (\mathbb{A}^{(\infty)})^*/K(M) & \xrightarrow[\sim]{j_1} & \text{spec}(\mathbb{Z}_{(p)}(\zeta_M))(\mathbb{C}) \end{array} \quad (7)$$

where $K = U_{\widehat{\mathbb{Z}}} \rtimes K(M)$ for the splitting (5). The morphism *can.* on the left hand side is given as follows:

$$[\alpha, \mu_{\log}, \rho] \mapsto [\alpha, \mu_{\log}, \mu_{\log} \alpha^{-1}(1), \rho]$$

where ρ must be chosen to be in $U_{\widehat{\mathbb{Z}}} \rtimes K(1)$ or even in $K(1) \cong \widehat{\mathbb{Z}}^*$ (this amounts to the same).

This yields a description of the canonical integral models $M({}^K\mathbf{X}), \Xi({}^K\mathbf{X})$ and $M({}^{K(M)}\mathbf{H}_0)$, respectively.

7.14. The lower morphisms p describe in all diagrams obvious structures of a relative group object, which are the same as those induced by the unipotent extension morphism of Shimura data $\mathbf{H}_0[U, 0] \rightarrow \mathbf{H}_0$, which is a group object (with the unit section given by the splitting γ , chosen above). The trivialization *can.* extends to partial compactifications, which are given by torus embeddings of $\mathbb{G}_{m, \mathbb{Z}_{(p)}[\zeta_M]} \otimes U_{\mathbb{Z}}$ over $\text{spec}(\mathbb{Z}_{(p)}[\zeta_M])$.

More precisely: By the arithmeticity condition and left invariance under conjugation by $P_{\mathbf{X}}(\mathbb{Q})$ a rational (w.r.t. $U_{\mathbb{Q}}$) polyhedral cone decomposition of $U_{\mathbb{R}}$ into cones $\{\sigma\}$ determines a rational polyhedral cone decomposition Δ of the conical complex (here $\mathbf{H}_0 \times U_{\mathbb{R}}(-1)$). On $\alpha \times U_{\mathbb{R}}(-1)$, the decomposition is given by $\{\alpha^{-1}(\sigma)\}$.

Hence, for each such σ the corresponding torus embedding is

$$\text{spec}(\mathbb{Z}_{(p)}[\zeta_M][U_{\mathbb{Z}}^*]) \hookrightarrow \text{spec}(\mathbb{Z}_{(p)}[\zeta_M][U_{\mathbb{Z}}^* \cap \sigma^{\vee}]).$$

They are glued in the usual way using the topology of the decomposition.

Proof of the Proposition. We begin by describing the horizontal morphisms in the diagram (6):

i_1 : Up to isomorphism any element in the category

$$[S^{-M} \mathbf{H}_0\text{-}\mathbb{Z}\text{-mot}]$$

is of the form $X = (\mathbb{G}_{m, S}, 0, \text{id}, \xi)$ with $\xi : (\mathbb{Z}/N\mathbb{Z})_S \rightarrow \mathbb{G}_m[M]_S \cong \text{spec}(\mathbb{Z}_{(p)}(\zeta_M))_S$. $i_1(X)$ is the morphism determined by $1 \in \mathbb{Z}/N\mathbb{Z}$, which determines the level-structure uniquely.

i_2 : Up to isomorphism any element in the category

$$[S^{-(M,0)} \mathbf{H}_0[U_{\mathbb{Z}}, 0]\text{-}L_{\mathbb{Z}}\text{-mot}]$$

is of the form $X = (\mu : (U_{\mathbb{Z}})_S \rightarrow \mathbb{G}_{m, S}, \text{id}, \text{id}, \xi)$. i_2 is defined as the fibre product of $i_1(X)$ (defined as before) with μ .

i_3 : Up to isomorphism any element in the category

$$[S^{-(M,0)} \mathbf{H}_0[U_{\mathbb{Z}}, 0]\text{-}L_{\mathbb{Z}}\text{-mot-triv}]$$

is of the form $X = (\mu : (U_{\mathbb{Z}})_S \rightarrow \mathbb{G}_{m, S}, \text{id}, \text{id}, \xi, \gamma)$, where $\gamma : L_S \rightarrow H_1^{dR}(M)$ is a trivialization (compatible with the $P_{\mathbf{X}, S}$ -structure). We have a canonical

trivialization $can. : F^0(H^{dR}(M)) \oplus \mathcal{O}_S(\frac{dz}{z})^* \cong H^{dR}(M)$, where $\mathcal{O}_S(\frac{dz}{z})^*$ is $\text{Lie}(\mathbb{G}_m)$ and $U_S^* \cong F^0(H^{dR}(M))$ (canonically via ρ) over any S . Mapping 1 to $(\frac{dz}{z})^*$ the domain of $can.$ can be identified with L_S . In other words, we have

$$can. : L_S \rightarrow H^{dR}(M).$$

Equipping a point

$$X = (\mu : (U_{\mathbb{Z}})_S \rightarrow \mathbb{G}_{m,S}, \text{id}, \text{id}, \xi) \in [S^{-(M,0)} \mathbf{H}_0[U_{\mathbb{Z}}, 0] \text{-} L_{\mathbb{Z}} \text{-mot}]$$

with the trivialization $can.$ defines the vertical morphism $can.$ on the left in (6). The morphism $can.$ on the right hand side is the obvious trivialization. The morphism i_3 is defined as the only $P_{\mathbf{X},S}$ -equivariant morphism that makes the diagram commutative (it maps X to $(can.^{-1} \circ \gamma, i_2(X)) \in P_{\mathbf{X}}(S) \times (\mathbb{G}_{m, \mathbb{Z}_{(p)}[\zeta_M]} \otimes U_{\mathbb{Z}}(S))$).

We now describe the resulting horizontal morphisms in diagram (7):

j_1 : We choose an isomorphism

$$\beta : H^B(\mathbb{G}_m, \mathbb{Z}) \rightarrow \mathbb{Z}.$$

Since $H_1^B(\mathbb{G}_m, \mathbb{Z}) = \mathbb{Z}(1)$, we have an $\alpha \in \mathbb{D}_{\mathbf{H}_0}$ with $\alpha^{-1} = \beta$.

Let ξ be a level structure. Its image in $\widehat{\mathbb{Z}}^*$ is given mod $K(M)$ by the composite (see 7.5):

$$\mathbb{Z}/N\mathbb{Z} \xrightarrow{\xi} H_1^{et}(\mathbb{G}_m, \mathbb{Z}/N\mathbb{Z}) = \mathbb{G}_m[N](\mathbb{C}) \xrightarrow{\rho_{et,B}} H_1^B(\mathbb{G}_m, \mathbb{Z}/N\mathbb{Z}) \xrightarrow{\beta} \mathbb{Z}/N\mathbb{Z}.$$

Note that $\rho_{et,B}$ is given by the logarithm.

We get an element

$$[\alpha, \rho] \in \mathbb{Q}^* \backslash \mathbb{D}_{\mathbf{H}_0} \times \mathbb{A}^{(\infty)*} / K(M)$$

where $\rho \in K(1)$ and have

$$j_1([\alpha, \rho]) = \exp(\alpha(\frac{1}{M}\rho)) \in \mathbb{G}_m(\mathbb{C})[M].$$

j_2 : Choose an isomorphism $\beta : H^B(M) \rightarrow \mathbb{Z} \oplus U_{\mathbb{Z}}^*$ (in the $P_{\mathbf{X}}(\mathbb{Z})$ -structure) such that $W_{-1}(\beta) = \alpha^{-1}$ as morphism $H_1^B(\mathbb{G}_m) = \mathbb{Z}(1) \rightarrow \mathbb{Z}$.

Recall that $L_{\mathbb{Z}} = \mathbb{Z} \oplus U_{\mathbb{Z}}^*$. The composite

$$L_{\mathbb{C}} \xrightarrow{can.} H^{dR}(M) \xrightarrow{\rho_{dR,B}} H^B(M) \otimes \mathbb{C} \xrightarrow{\beta} L_{\mathbb{C}}.$$

maps $(1, 0)$ to $(\alpha^{-1}(1), 0)$ and $(0, u^*)$ to $(-\alpha^{-1}(\log(\mu(u^*))), u^*)$. This is the same as the operation of $\mu_{\log} \alpha^{-1}(1) \in P_{\mathbf{X}}(\mathbb{C})$ under the action defined in 7.3, 4., where $\mu_{\log} \in U_{\mathbb{C}}/U_{\mathbb{Z}}$ is such that $u^*(\mu_{\log}) = \alpha^{-1}(\log(\mu(u^*)))$ for all $u^* \in U_{\mathbb{Z}}^*$. We have then $\beta(F^0(M)) = \mu_{\log} U_{\mathbb{C}}^*$.

Hence the moduli point of $([\mu : U_{\mathbb{Z}}^* \rightarrow \mathbb{G}_m], \text{id}, \text{id}, \xi)$ in the complex analytic Shimura variety is:

$$[\alpha, \mu_{\log}, \rho] \in P(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) / K$$

j_3 : The image of the point $[\alpha, \mu_{\log}, \rho]$ via *can.* in the standard principal bundle is given by

$$[\alpha, \mu_{\log}, \mu_{\log} \alpha(1)^{-1}, \rho] \in P(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P_{\mathbf{X}}(\mathbb{C}) \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K.$$

Therefore the trivialization *can.* on the left in diagram (7) is given by

$$\begin{aligned} P(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P(\mathbb{A}^{(\infty)})/K &\rightarrow P(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P(\mathbb{C}) \times P(\mathbb{A}^{(\infty)})/K \\ [\alpha, \mu_{\log}, \rho] &\mapsto [\alpha, \mu_{\log}, \mu_{\log} \alpha(1)^{-1}, \rho] \end{aligned}$$

for $\rho \in U_{\mathbb{Z}} K(1)$ (!), as claimed. This also defines the morphism j_3 by $P_{\mathbf{X}}(\mathbb{C})$ -equivariance.

It is easy to see that this morphism *can.* is compatible with the action of the torus multiplication in the relative torus $M^{(K)\mathbf{X}}$ over $M^{(K(N))\mathbf{H}_0}$. It is hence by the characterization 6.14, 3. the trivialization which extends to every partial compactification $M^{(K)}_{\Delta} \mathbf{X}$.

The model of the Shimura variety (resp. standard principal bundle) described by the schemes representing

$$[{}^K \mathbf{H}_0[U, 0]\text{-}L\text{-}\mathbf{mot}], \text{ resp. } [{}^K \mathbf{H}_0[U, 0]\text{-}L\text{-}\mathbf{mot-triv}]$$

are the canonical ones by 7.6 and 7.9. \square

8. q -expansion principle

8.1. From the abstract theory of canonical models of toroidal compactifications as described by the Main Theorems 6.9 and 6.14, a classical “ q -expansion principle” can be derived. In this section we explain this for a wide class of Shimura varieties with boundary.

8.2. Resume the discussion from 7.12. Let \mathcal{E} be a $P_{\mathbf{X}}$ -equivariant vector bundle on $M^{\vee}(\mathbf{X}) = \mathbb{W}(U)v$. Here v is the $\mathbb{Z}_{(p)}$ -valued point corresponding to the splitting (5). $H^0(M^{(K)\mathbf{X}}, \Xi^* \mathcal{E})$ is identified with the $P_{\mathbb{Q}}$ -invariant functions on

$$\mathbb{D}_{\mathbf{X}} \times P(\mathbb{A}^{(\infty)})/K \rightarrow \mathcal{E}$$

Choose a basis v_1, \dots, v_n of the fibre \mathcal{E}_v .

The canonical trivialization (*can.* of the Proposition 7.13) of the standard principal bundle trivializes also (the algebraic) $\Xi^* \mathcal{E}$. For this consider the following diagram

$$M^{(K)\mathbf{X}} \xrightleftharpoons[p]{can.} P^{(K)\mathbf{X}} \xrightarrow{\Pi} M^{\vee}(\mathbf{X}),$$

$\Xi^* \mathcal{E}$ is defined as the P -invariants in the pullback $\Pi^* \mathcal{E}$. *can.* identifies those with $can.^* \Pi^* \mathcal{E} = (\mathcal{E}_v)_{M^{(K)\mathbf{X}}}$. This last identification comes from the fact, that

$\Pi \circ \text{can.}$ is constant with image v by construction. By Proposition 7.13 this trivialization, base changed to \mathbb{C} , is given by the function

$$\begin{aligned} s_i : \mathbb{D}_{\mathbf{H}_0} \times U_{\mathbb{C}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K &\rightarrow \mathcal{E} \\ [\alpha, \mu_{\log}, \rho] &\mapsto \mu_{\log} \alpha(1)^{-1} v_i \quad \text{for } \rho \in K_U K(1), \end{aligned}$$

where $\mathfrak{B} = \{v_i\}$, and the v_i are considered as points in the fibre over v . (Extend the function to other ρ by requiring $P_{\mathbf{X}}(\mathbb{Q})$ -invariance).

A general section may be written as

$$[\alpha, \mu_{\log}, \rho] \mapsto \sum_{k \in U_{\mathbb{Z}}^*, i} a_{k, \alpha, \rho, i} \exp(\alpha(\mu_{\log} k)) s_i$$

($\rho \in K(1) = \widehat{\mathbb{Z}}^*$), and the corresponding i -th coordinate-function on $\mathbb{G}_{m, \mathbb{C}} \otimes U_{\mathbb{Z}} \times_{\mathbb{C}} \mathbb{C}[\zeta_M] = \text{spec}(\bigoplus_{\zeta_M, \mathbb{C}} \mathbb{C}[U_{\mathbb{Z}}^*])$ is given by

$$\sum_{k \in U_{\mathbb{Z}}^*} a_{k, \alpha, \rho, i} [k]$$

in the fibre above $\zeta_{M, \mathbb{C}} = \exp(\alpha(\frac{1}{M}\rho))$.

Let σ now be a maximal dimensional rational polyhedral cone of $U_{\mathbb{R}}$. The completion at the corresponding boundary stratum is given by

$$\text{spf}(\bigoplus_{\zeta_{M, \mathbb{C}}} \mathbb{C}[[U_{\mathbb{Z}}^* \cap \sigma^{\vee}]]).$$

If a formal function f given by

$$\sum_{k \in U_{\mathbb{Z}}^* \cap \sigma^{\vee}} a_{k, \alpha, \rho}(f) [k]$$

in the fibre above the respective $\zeta_{M, \mathbb{C}} = \exp(\alpha(\frac{1}{M}\rho))$ converges in a neighborhood of the boundary points, it has the form

$$[\alpha, \mu_{\log}, \rho] \mapsto \sum_{k \in U_{\mathbb{Z}}^*} a_{k, \alpha, \rho}(f) \exp(\alpha(\mu_{\log} k)),$$

there, too.

We make the following observation which is important for the q -expansion principle: The function ${}^{\tau}f$ for an automorphism $\tau \in \text{Aut}(\mathbb{C})$ has the expansion

$$\sum_{k \in U_{\mathbb{Z}}^* \cap \sigma^{\vee}} {}^{\tau}a_{k, \alpha, \rho}(f) [k]$$

in the fibre over ${}^{\tau}(\zeta_{M, \mathbb{C}}) = \exp(\alpha(\frac{1}{M}\rho\kappa_{\tau}))$. Here $\kappa_{\tau} \in K(1)/K(M)$ is determined by this (image of τ under the class field theory isomorphism).

Therefore the Fourier coefficients satisfy:

$$a_{k, \alpha, \rho}({}^{\tau}f) = {}^{\tau}a_{k, \alpha, \rho\kappa_{\tau}^{-1}}(f). \quad (8)$$

8.3. We continue by stating a formal q -expansion principle, stating roughly that — under certain hypotheses — a regular function is integral, if after completion at some point, the corresponding formal function is integral.

For a commutative ring R and ideal $I \subseteq R$, we denote by $C_I(R)$ the completion of R w.r.t. the I -adic topology. For an I -adically complete ring R and a multiplicatively closed subset $S \subset R$, we denote by $R\{S^{-1}\}$ the completion w.r.t. $I[S^{-1}]$ of the localization $R[S^{-1}]$.

Let R be an excellent normal integral domain and I an ideal, such that $\text{spf}(C_I(R))$ is connected (or equivalently, such that R/\sqrt{I} has no nontrivial idempotents). Let s be a prime element of R , neither a zero divisor nor a unit of R/I . Let M be a projective R -module.

Lemma 8.4. *The diagram*

$$\begin{array}{ccc} M^\wedge & \xrightarrow{\quad} & M[s^{-1}] \\ \downarrow & & \downarrow \\ C_I(M)^\wedge & \xrightarrow{\quad} & C_I(M)\{s^{-1}\} = C_{I[s^{-1}]}(M[s^{-1}]) \end{array}$$

is Cartesian.

Proof. Because M is projective this follows from the case $M = R$, which we assume from now on.

The top horizontal map is injective because R is integral. The vertical maps are injective by Krull's Theorem. Because of the assumptions, the ring $C_I(R)$ is an integral domain. Hence the bottom horizontal arrow is injective. Therefore also the map $C_I(R)[s^{-1}] \rightarrow C_I(R)\{s^{-1}\}$ is injective, and we are left to show that for arbitrary n , the left square in the following diagram with exact lines is Cartesian:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & \frac{1}{s^n}R & \xrightarrow{\cdot s^n} & R/s^n R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{R} & \longrightarrow & \widehat{\frac{1}{s^n}R} & \longrightarrow & \widehat{R/s^n R} \longrightarrow 0 \end{array}$$

where in the bottom line we mean the I -adic completions of the respective f.g. R -modules. (The rows are exact because exactness of the completion functor on f.g. modules. This is, however, not used in the sequel.) We have to show that the right vertical map is injective. This is the case if $0 \in R/s^n R$ is closed w.r.t. the I -adic topology. Its closure is (by an extension of Krull's Theorem [5, Chap. III, §3, 2., Prop. 5]) the set of $x \in R/s^n R$ for which there exists an $m \in I$, such that $(1-m)x = 0$ holds true. Therefore the above map is injective, if no $m \in I$ and $x \in R$ exist, such that $s^n | (1-m)x$ and $s^n \nmid x$. Since s is prime, this is the case, if no $m \in I$ exists, such that $s | 1-m$. Since by assumption s is not a unit modulo I , this is indeed impossible. \square

8.5. Let ${}^K_{\Delta}\mathbf{X}$ be p -ECMSD such that an integral canonical model $M({}^K_{\Delta}\mathbf{X})$ exists. We consider Δ “up to refinements” in this section, i.e. we feel free to refine it whenever necessary.

Assume that there is a boundary morphism:

$$\iota : \mathbf{H}_0[U, 0] \Longrightarrow \mathbf{X},$$

which we fix once and for all in this section.

By [36, Proposition 12.1], its existence forces the reflex field of \mathbf{X} to be \mathbb{Q} . We write also shorthand $\mathbf{B} = \mathbf{H}_0[U, 0]$. It induces a dual boundary morphism (open embedding):

$$M^{\vee}(\iota) : M^{\vee}(\mathbf{B}) \hookrightarrow M^{\vee}(\mathbf{X}).$$

between $\mathbb{Z}_{(p)}$ -schemes and therefore a base point $M^{\vee}(\iota)(v) \in M^{\vee}(\mathbf{X})(\mathbb{Z}_{(p)})$ which we denote by v , too, ι being fixed.

Let \mathcal{E} be a $P_{\mathbf{X}}$ -equivariant line bundle on $M^{\vee}(\mathbf{X})$.

For any $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ we have an associated boundary morphism

$$(\iota, \rho) : {}^{K_1}_{\Delta_1}\mathbf{B} \Longrightarrow {}^K_{\Delta}\mathbf{X},$$

where $K_1 = K^{\rho} \cap \iota(P_{\mathbf{B}}(\mathbb{A}^{(\infty)}))$ and Δ_1 is defined as pullback (3.21). Refining the original Δ , if necessary, we can assume that Δ_1 is smooth.

$M(\iota, \rho)^* \Xi^* \mathcal{E} \cong \Xi^* M^{\vee}(\iota)^* \mathcal{E}$ (6.14, 1.) may be trivialized as in 8.2, using a basis vector z' of $(M^{\vee}(\iota)^* \mathcal{E})_v = \mathcal{E}_v$. K_1 always contains a group of the form $U_{\mathbb{Z}} K(M)$ for a saturated (at p) lattice $U_{\mathbb{Z}} \subseteq U$ and $p \nmid M$.

Let $s_{z'}$ denote the corresponding trivializing section (8.2). Thus every section $f \in H^0(M({}^K_{\Delta}\mathbf{X})_{\mathbb{C}}, \Xi^* \mathcal{E})$ yields Fourier coefficients (see 8.2):

$$a_{\iota, z', k, \alpha, \rho}(f) := a_{k, \alpha, 1} \left(\frac{M(\iota, \rho)^* f}{s_{z'}} \right).$$

This definition includes all the Fourier expansions defined before because we have for $\rho' \in \widehat{\mathbb{Z}}^* = K(1) \subseteq \mathbb{A}^{(\infty)*}$:

$$a_{\iota, z', k, \alpha, \rho \rho'}(f) = a_{k, \alpha, \rho'} \left(\frac{M(\iota, \rho)^* f}{s_{z'}} \right).$$

Lemma 8.6. *If $f \in H^0(M({}^K_{\Delta}\mathbf{X})_{\mathbb{C}}, \Xi^* \mathcal{E}_{\text{merom.}})$ is a meromorphic section (any Δ , including $\{0\}$ — the meromorphic sections are the same), we get Fourier coefficients*

$$a_{\iota, z', k, \alpha, \rho, \sigma}(f) := a_{k, \alpha, \rho, \sigma} \left(\frac{M(\iota, \rho)^* f}{s_{z'}} \right),$$

which are defined the same way (8.2) but valid analytically only on a set of the form:

$$\{\alpha\} \times (x + U_{\mathbb{R}} + \sigma) \times \{\rho\} \subset \mathbb{D}_{\mathbf{B}} \times P_{\mathbf{B}}(\mathbb{A}^{(\infty)}),$$

where σ is a maximal dimensional rational polyhedral cone in $U_{\mathbb{R}}(-1)$.

Proof. We show that there is a boundary point on *some* toroidal compactification over \mathbb{C} at which locally the divisor of f consists only of components of the boundary divisor itself. From this the statement follows because a small neighborhood around such a boundary point is mapped to the set given in the statement of the lemma.

It suffices to check this condition in a formal neighborhood of the boundary point. Writing f as a quotient of formal functions, we are reduced to the case, where f is a formal function in the formal completion at some boundary point associated with $\iota, \rho, \alpha \in \mathbb{D}_{\mathbf{H}_0}$ and a r.p.c. σ of maximal dimension (now considered in $U_{\mathbb{R}}$, see 7.14). Let the cone be generated by $\{\lambda_i \in U_{\mathbb{Z}}\}$ and call $\{z_i\}$ the variables corresponding to the vertices of the dual cone. This formal completion is

$$\mathrm{spf}\left(\bigoplus_{\zeta_{M,\mathbb{C}}} \mathbb{C}[[U_{\mathbb{Z}}^* \cap \sigma^\vee]]\right).$$

Hence the formal function is an element in

$$\bigoplus_{\zeta_{M,\mathbb{C}}} \mathbb{C}[[U_{\mathbb{Z}}^* \cap \sigma^\vee]] = \bigoplus_{\zeta_{M,\mathbb{C}}} \mathbb{C}[[z_1, \dots, z_n]].$$

We consider its components separately and call f any of them.

We may now refine the polyhedral cone decomposition in a way that it induces barycentric subdivisions of faces of σ . The faces τ of σ correspond to subsets $S \subseteq \{1, \dots, n\}$ (the vertices of τ). There are then $|S|$ new maximal dimensional cones σ' in the corresponding subdivision, indexed by the respective vertex j which does not belong to σ' anymore. Let $\lambda'_1, \dots, \lambda'_n$ be the generators of the corresponding cone. We have $\lambda'_j = \sum_{i \in S} \lambda_i$ and $\lambda'_i = \lambda_i$, $i \neq j$. We get a corresponding homomorphism:

$$\mathbb{C}[[z_1, \dots, z_n]] \rightarrow \mathbb{C}[[z'_1, \dots, z'_n]].$$

given by $z_j \mapsto \sum_{i \in S} z'_i$ and $z_i \mapsto z'_i$, $i \neq j$.

Since both rings have the same form, we may consider this homomorphism as automorphism $T_{j,S}$ of the ring on the left, in the obvious way. We have to show that repeatedly applying those to f and clearing factors of z_i , $i = 1, \dots, n$ one arrives at an element \tilde{f} containing a constant term. Let S_j be the product $T_{i, \{i, j\}}$ for $i \neq j$. It is given by mapping $z_j \mapsto z_j$ and $z_i \mapsto z_i z_j$ for $i \neq j$.

We show by induction on the minimal degree of the occurring monomials in f , that by repeatedly applying the S_i to f and clearing factors of z_i one arrives at a function \tilde{f} containing a constant term. Let $N = \sum_i n_i$ be the minimal degree of a monomial $g = \prod z_i^{n_i}$ occurring in f . Let j be such that $n_j \neq 0$. By applying S_j to f , a monomial $h = \prod z_i^{m_i}$ gets transformed to $z_j^{\deg(h)} \prod_{i \neq j} z_i^{m_i}$. Since $\deg(g)$ is minimal, we may define \tilde{f} by $\frac{S_j(f)}{z_j^{\deg(g)}}$. We have $\deg(\frac{S_j(f)}{z_j^{\deg(g)}}) = \sum_{i \neq j} n_i < \deg(g)$, hence \tilde{f} has a term of smaller minimal degree. \square

Theorem 8.7 (*q-expansion principle*). *Assume 6.6.*

With the notations of 8.5, there is an $M \in \mathbb{N}$, $p \nmid M$ and a finite subset $\{\rho_1, \dots, \rho_n\} \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, such that for $R := \mathbb{Z}_{(p)}[\zeta_N]$, $M|N$, we have

$$H^0(M(\frac{K}{\Delta}\mathbf{O})_R, \Xi^*\mathcal{E}) = \{f \in H^0(M(\frac{K}{\Delta}\mathbf{O})_{\mathbb{C}}, \Xi^*\mathcal{E}) \mid \forall i \ \forall k \in U_{\mathbb{Q}}^*, \forall \alpha \ a_{\iota, z', k, \alpha, \rho_i}(f) \in R\}$$

and similarly⁶

$$H^0(M(\frac{K}{\Delta}\mathbf{O})_R, \Xi^*\mathcal{E}_{horz}) = \left\{f \in H^0(M(\frac{K}{\Delta}\mathbf{O})_{\mathbb{C}}, \Xi^*\mathcal{E}_{merom.}) \mid \begin{array}{l} \forall i \ \forall \alpha \ \exists \sigma \subset U_{\mathbb{R}} \text{ open r.p. cone,} \\ \forall k \in U_{\mathbb{Q}}^*, \ a_{\iota, z', k, \alpha, \rho_i, \sigma}(f) \in R \end{array} \right\}.$$

Moreover, we have

$$a_{\iota, z', k, \alpha, \rho, \sigma}(\tau f) = \tau(a_{\iota, z', k, \alpha, \rho k_{\tau}^{-1}, \sigma}(f)) \quad \forall \tau \in \text{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}),$$

hence f is (in both cases) defined over $\mathbb{Z}_{(p)}$ instead of R , if it satisfies

$$a_{\iota, z', k, \alpha, \rho, \sigma}(f) = \tau(a_{\iota, z', k, \alpha, \rho k_{\tau}^{-1}, \sigma}(f)),$$

for all $\tau \in \text{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q})$, all ρ, α, k and some σ (depending on ρ and α). Here k_{τ} is the image of τ under the natural isomorphism $\text{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}) \cong K(1)/K(N)$.

Proof. We may show the theorem for a particular Δ and all of its refinements. A posteriori, it will then be true for any Δ' , using a common refinement of Δ and Δ' . Let f be a complex section of the bundle $\Xi^*(\mathcal{E})$ over $M(\frac{K}{\Delta}\mathbf{X})$. By the abstract q -expansion principle 8.4, a section of a locally free sheaf is defined over $M(\frac{K}{\Delta}\mathbf{X})_R$ if on any connected component (over R), there is a point p , defined over R , such that after passing to the completion at p , the corresponding formal series is defined over R . Now on any connected component i of $M(\frac{K}{\Delta}\mathbf{X})_{\mathbb{C}}$, there is a boundary point corresponding to a boundary component (ι, ρ_i) and a $\sigma \subset U_{\mathbb{R}}$. Using the boundary isomorphism $M(\iota, \rho_i)$ (6.9), defined over $\mathbb{Z}_{(p)}$, it suffices to show that the pullback of f to $\Xi^*(M^{\vee}(\iota)^*\mathcal{E})$ over $M(\widehat{\frac{K'_i}{\Delta_i}\mathbf{H}_0}[U, 0])$ (completion with respect to the boundary point corresponding to σ) is defined over R . We may choose the integer M such that all connected components of $M(\frac{K'_i}{\Delta_i}\mathbf{H}_0[U, 0])$ are defined over R , for all i and such that we may test f by pulling it further back to $M(\widehat{\frac{K'_i}{\Delta_i}\mathbf{H}_0}[U, 0])$, where $K'_i = U_{\mathbb{Z}, i}K(M)$ for some lattice $U_{\mathbb{Z}, i} \subset U$ as in 8.2. We refine Δ such that Δ_i is smooth w.r.t. K'_i for all i . By the consideration in 8.5 above, the formal function $f/s_{z'}$ on

$$M(\widehat{\frac{U_{\mathbb{Z}, i}K(M)}{\Delta_i}\mathbf{H}_0}[U, 0])$$

⁶*horz.* means that we inverted all functions, which have no component of the fibre above p in its divisor.

for $U_{\widehat{\mathbb{Z}}, i} K(M) \subseteq K_i$ is given by

$$\left(\frac{f}{s}\right)_{\zeta_{M, \mathbb{C}}} = \sum_{\lambda \in U_{\widehat{\mathbb{Z}}}^*} \alpha_{\iota, z', k, \alpha, \rho_i k, \lambda, \sigma}(f).$$

Here $[\alpha, k] \in \mathbb{Q}^* \backslash \mathbb{D}_{\mathbf{H}_0} \times \mathbb{A}^{(\infty)*} / K(M)$, with $k \in K(1)$ is such that $\zeta_M = \exp(\alpha(\frac{k}{M}))$. (It suffices of course to look at the fibre over the $\zeta_{M, \mathbb{C}}$ corresponding to $[\alpha, 1]$). Then, since by assumption these coefficients lie in R , $f/s_{z'}$ is defined over R and hence f itself. The first statements follow because we investigated at least one boundary point on each connected component. If f was a meromorphic section, we may refine Δ in a way such that for all i its pullback Δ_i contains a cone determined by $\sigma \subset U(\mathbb{R})$ such that there exists a Fourier expansion in a neighborhood of the corresponding boundary point (see 8.6). The statement about the Fourier coefficients of the Galois conjugates was shown in 8.2. \square

9. Shimura varieties of orthogonal type, special cycles

Proposition 9.1. *If $\dim(L) \geq 3$, the reflex field of $\mathbf{S}(L)$ is \mathbb{Q} . The compact dual is the zero quadric, i.e.*

$$\mathbf{M}^\vee(\mathbf{S}(L))(R) = \{ \langle z \rangle \in \mathbb{P}(L_R) \mid \langle z, z \rangle = 0 \}$$

for any ring R over $\mathbb{Z}_{(p)}$.

If $\dim(L) = 2$, the reflex field of $\mathbf{S}(L)_+$, resp. $\mathbf{S}(L)_-$ is the imaginary quadratic field (in \mathbb{C}) associated with the (negative) definite binary quadratic form L . The compact dual is a point given by the isotropic vector corresponding to the filtration in $\mathbb{D}_{\mathbf{S}(L)_+}$, respectively $\mathbb{D}_{\mathbf{S}(L)_-}$.

Proof. Let D be a negative definite subspace D defined over $\mathbb{Z}_{(p)}$. Over an imaginary quadratic field F , unramified at p , there is a morphism

$$\mu : \mathbb{G}_m \hookrightarrow \mathrm{GSpin}(L),$$

acting with exponent 2 on one isotropic vector z , with exponent 0 on \bar{z} and with exponent 1 on the orthogonal complement. Over \mathbb{C} it yields the associated morphism u_h (6.2), where h is one of the 2 morphisms corresponding to D . The morphism $\overline{u_h}$ is also of the form $u_{h'}$ and hence conjugated to u_h if $n \geq 3$. The conjugacy class of these morphisms is therefore defined over \mathbb{Q} . If $n = 2$, $\mathrm{GSpin}(L)$ is a torus, and hence the reflex field is F . Furthermore, the morphism u defines the filtration $0 \subset \langle z \rangle \subset \langle z \rangle^\perp \subset L$ which is completely determined by $\langle z \rangle$. \square

Remark 9.2. *The Hodge embedding from Lemma 5.4 induces a morphism (cf. Main Theorem 6.12)*

$$\mathbf{M}^\vee(\mathbf{S}(L)) \hookrightarrow \mathbf{M}^\vee(\mathbf{H}(\mathbf{C}^+(L), \langle \cdot, \cdot \rangle_\delta)).$$

Here $M^\vee(\mathbf{S}(L))$ is the space of isotropic lines in L and $M^\vee(\mathbf{H}(C^+(L), \langle \cdot, \cdot \rangle_\delta))$ is the space of Lagrangian saturated sublattices of $C^+(L)$ with respect to the form $\langle \cdot, \cdot \rangle_\delta$. The proof of Lemma 5.4 shows that this map is given (for any R) by sending $a < z >$ in L_R , with z assumed to be primitive, to the subspace $zwC^+(L_R)$, where w is a primitive vector with $\langle z, w \rangle = 1$ (this subspace actually does not depend on w). One also immediately sees that this is compatible with the Borel embedding because $w = \frac{\bar{z}}{\langle z, \bar{z} \rangle}$ leads to the projector considered in the proof of Lemma 5.4.

Lemma 9.3. *Let K be a neat admissible compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}(\infty)})$ and Δ be a smooth K -admissible rational polyhedral cone decomposition. The (geometric) connected components of $M_{\Delta}^{(K)\mathbf{O}}_{\overline{\mathbb{Q}}}$ and $M_{\Delta}^{(K)\mathbf{O}}_{\overline{\mathbb{F}}_p}$ are in bijection and defined over $\mathrm{spec}(\mathbb{Z}_{(p)}[\zeta_N])$, where ζ_N , $p \nmid N$ is an N -th root of unity.*

Proof. This follows directly from 6.5, 6.9 using [Milne4, Theorem 5.17]. \square

Lemma 9.4. *Let $L_{\mathbb{Z}}$ be a lattice with quadratic form of discriminant $D \neq 0$. Let K be a compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}(\infty)})$, and pick a set of representatives*

$$g_i \in \mathrm{SO}(\mathbb{Q}) \backslash \mathrm{SO}(\mathbb{A}^{(\infty)}) / K.$$

For each g_i there is the lattice $L_{\mathbb{Z}}^{(i)}$ characterized by $L_{\mathbb{Z}_p}^{(i)} = g_i L_{\mathbb{Z}_p}$. We have

$$\left[\mathrm{SO}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{O}} \times (\mathrm{SO}(\mathbb{A}^{(\infty)}) / K) \right] = \bigcup_i [\Gamma_i \backslash \mathbb{D}_{\mathbf{O}}],$$

where Γ_i is the subgroup of $\mathrm{SO}(L_{\mathbb{Z}}^{(i)})$ defined by $\mathrm{SO}(\mathbb{Q}) \cap g_i K g_i^{-1}$. If K is the discriminant kernel of $L_{\mathbb{Z}}$, each Γ_i is the discriminant kernel of the respective $L_{\mathbb{Z}}^{(i)}$.

Proof. [H, 1.10] or [Milne4]. \square

9.5. Let $L_{\mathbb{Z}_{(p)}}$ be a lattice with unimodular quadratic form, i.e. inducing an isomorphism $L_{\mathbb{Z}_{(p)}} \cong (L_{\mathbb{Z}_{(p)}})^*$, and $L'_{\mathbb{Z}_{(p)}}$ be a saturated sublattice with unimodular form, of signature $(m-2, 2)$ and $(m-2-n, 2)$, respectively, where $0 < n \leq m-2$.

Recall the embedding 5.4, here used mod \mathbb{G}_m

$$\iota : \mathbf{O}(L') \hookrightarrow \mathbf{O}(L),$$

in the case $m-n > 2$ and the 2 embeddings

$$\iota_{\pm} : \mathbf{O}(L')_{\pm} \hookrightarrow \mathbf{O}(L),$$

in the case $m-n = 2$.

Let an admissible compact open subgroup K be given. For each $g \in \mathrm{SO}(L_{\mathbb{A}(\infty, p)})$ we get a conjugated embedding:

$$(\iota, g) : {}^{K'_g} \mathbf{O}(L') \hookrightarrow {}^K \mathbf{O}(L)$$

for $K'_g = K \cap g^{-1}\iota(\mathrm{SO}(L'_{\mathbb{A}(\infty)}))g$.

In addition, we may find smooth (w.r.t. K , resp. K'_g) complete and projective Δ , and Δ'_g for each g (3.22), such that models exist (6.9), and we have, in the end, embeddings of p -ECMSD:

$$(\iota, g) : {}^{K'_g}_{\Delta'_g} \mathbf{O}(L') \hookrightarrow {}^K_{\Delta} \mathbf{O}(L)$$

(resp. with \pm).

Let $M_{\mathbb{Z}_{(p)}}$ be another lattice with unimodular quadratic form and define

$$I(M, L)_R := \{\alpha : M_R \rightarrow L_R \mid \alpha \text{ is an isometry}\}.$$

We have $I(M, L)_{\mathbb{Q}} \neq \emptyset \Leftrightarrow I(M, L)_{\mathbb{A}(\infty)} \neq \emptyset$ by Hasse's principle. If these sets are nonempty, we may then find even an $x \in I(M, L)_{\mathbb{Z}_{(p)}}$.

For a K -invariant admissible⁷ Schwartz function $\varphi \in S((M^* \otimes L)_{\mathbb{A}(\infty)})$

$$I(M, L)_{\mathbb{A}(\infty)} \cap \mathrm{supp}(\varphi) = \coprod_j K g_j^{-1} x$$

with finitely many $g_j \in \mathrm{SO}(L_{\mathbb{A}(\infty, p)})$. We assume that Δ has been refined such that Δ'_{g_j} exists, with the properties claimed above for all j .

Definition 9.6. *If $I(M, L)_{\mathbb{Q}} \neq \emptyset$, we define the p -integral **special cycle** associated with $M_{\mathbb{Z}_{(p)}}$ and φ on $M({}^K_{\Delta} \mathbf{O}(L))$ as*

$$Z(L, M, \varphi; K) := \sum_j \varphi(g_j^{-1} x) \mathrm{im}(M(\iota, g_j)),$$

where ι is associated with the embedding $x^{\perp} \hookrightarrow L_{\mathbb{Z}_{(p)}}$. If $I(M, L)_{\mathbb{Q}} = \emptyset$ we define $Z(L, M, \varphi; K) := 0$.

Note that if $m - n = 2$, $M({}^{K'_g}_{\Delta'_g} \mathbf{O}(x^{\perp}))$ is actually equipped with a morphism to $\mathrm{spec}(\mathcal{O}_{\mathbf{X}})$ which is an extension of $\mathrm{spec}(\mathbb{Z}_{(p)})$. In the definition above, we consider, however, the structural morphism to $\mathrm{spec}(\mathbb{Z}_{(p)})$. (For the \mathbb{C} -points this means, that we get in any case the special cycles as defined e.g. in [3] or [22].) The sum is finite and independent of the choice of the g_i . The formation is compatible with Hecke operators (3.21) in the obvious sense.

If $I(M, L)_{\mathbb{Q}} \neq \emptyset$ and $M_{\mathbb{Q}}$ is only a \mathbb{Q} -vector space with non-degenerate quadratic form and φ any Schwartz function, we write

$$I(M, L)_{\mathbb{A}(\infty)} \cap \mathrm{supp}(\varphi) = \coprod_j K g_j^{-1} x$$

for any $x \in I(M, L)_{\mathbb{Q}}$.

⁷by this we mean that φ is the tensor product of a Schwartz function $\varphi \in S((M^* \otimes L)_{\mathbb{A}(\infty, p)})$ with the characteristic function of $(M^* \otimes L)_{\mathbb{Z}_p}$

Definition 9.7. We define the rational **special cycle** associated with $M_{\mathbb{Q}}$ and φ on $M(\Delta^K \mathbf{O}(L))$ as

$$Z(L, M, \varphi; K) := \sum_j \varphi(g_j^{-1}x) \overline{\text{im}(M(\iota, g_j))}^{Zar}.$$

Here ι is associated with the embedding $x^\perp \hookrightarrow L_{\mathbb{Q}}$ and $M(\iota, g_j)$ is only a morphism of rational Shimura varieties. Here even Δ may be chosen arbitrary. In that case we understand the Zariski closure of the embedding of the uncompactified rational Shimura variety.

From the extension property (cf. Main Theorem 6.5, 3.) and the properties of toroidal compactifications (cf. Main Theorem 6.9) follows that this definition coincides with the previous one, if there exists a lattice $M_{\mathbb{Z}_{(p)}}$, such that the restriction of the quadratic form is unimodular, φ is admissible, and Δ is chosen in such a way that we have embeddings of p -ECMSD.

10. The arithmetic theory of automorphic forms

10.1. Let $\Delta^K \mathbf{X}$ be pure p -ECMSD (3.21) with smooth (w.r.t. K), projective, and complete Δ , $E \subset \mathbb{C}$ be the reflex field and $\mathcal{O}_{\mathbf{X}} = \mathcal{O}_{E, (p)}$ the reflex ring. We have an isomorphism

$$M(\Delta^K \mathbf{X}) \times_{\text{spec}(\mathbb{Z}_{(p)})} \mathbb{C} = \coprod_{\sigma \in \text{Hom}(E, \mathbb{C})} M(\Delta^K \mathbf{X}) \times_{\mathcal{O}_{\mathbf{X}}, \sigma} \mathbb{C}.$$

Conjecture A of [32, p. 58] which is proven in [32, Corollary 18.5] states in particular that for each $\sigma \in \text{Hom}(E, \mathbb{C})$ there is a (rational) Shimura datum \mathbf{X}^σ , and we have an isomorphism

$$(M(\Delta^K \mathbf{X}) \times_{\mathcal{O}_{\mathbf{X}}, \sigma} \mathbb{C})^{an} \rightarrow [P_{\sigma \mathbf{X}} \backslash \mathbb{D}_{\sigma \mathbf{X}} \times P_{\sigma \mathbf{X}}(\mathbb{A}^{(\infty)}) / K] \quad (9)$$

which renders the system $M(\Delta^K \mathbf{X}) \times_{\mathcal{O}_{\mathbf{X}}, \sigma} {}^\sigma \mathcal{O}_{\mathbf{X}}$ with a twisted adelic action into a rational canonical model for the Shimura datum ${}^\sigma \mathbf{X}$.

The compact dual $M^\vee(\mathbf{X})$, considered as scheme over $\text{spec}(\mathbb{Z}_{(p)})$, can be identified also with a closed subscheme of \mathcal{QPAR} (2.8) and carries an action of $P_{\mathbf{X}}$.

Let \mathcal{E} a locally free sheaf on the stack (over $\text{spec}(\mathbb{Z}_{(p)})!$)

$$[M^\vee(\mathbf{X}) / P_{\mathbf{X}}]$$

(i.e. a $P_{\mathbf{X}}$ -equivariant sheaf on $M^\vee(\mathbf{X})$) and $h_E = \{h_{E, \sigma}\}$ for $\sigma \in \text{Hom}(E, \mathbb{C})$ be a collection of $P_{\sigma \mathbf{X}}(\mathbb{R})U_{\sigma \mathbf{X}}(\mathbb{C})$ -invariant Hermitian metric on $E_{\mathbb{C}}|_{h(\mathbb{D}_{\sigma \mathbf{X}})}$ where $\bigcup_{\sigma} h(\mathbb{D}_{\sigma \mathbf{X}})$ is embedded into $(M^\vee(\mathbf{X}) \times_{\mathbb{Z}_{(p)}} \mathbb{C})(\mathbb{C})$ via the Borel embeddings — 6.12 — taking into account the isomorphism (9).

Recall from 6.14 the 1-morphism

$$\Xi : M(\Delta^K \mathbf{X}) \rightarrow [M^\vee(\mathbf{X}) / P_{\mathbf{X}}].$$

10.2. Let D be the boundary divisor (6.9). It is a divisor of normal crossings. We will define a Hermitian metric Ξ^*h_E on $(\Xi^*\mathcal{E})_{\mathbb{C}}$, singular along $D_{\mathbb{C}}$, as follows⁸: We do this for each ${}^\sigma\mathbf{X}$ separately, so let \mathbf{X} be one of them. By definition $\Xi^*E_{\mathbb{C}}(U)$, for any open $U \subset M(\frac{K}{\Delta}\mathbf{X})(\mathbb{C})$, is given by the $P(\mathbb{Q})$ -invariant sections of the pullback of $E_{\mathbb{C}}$ to the standard principal bundle $P(\frac{K}{\Delta}\mathbf{X})$ (the right $P_{\mathbf{X}}$ -torsor describing the morphism Ξ) via the compatibility isomorphism of Ξ over \mathbb{C} with $\Xi_{\mathbb{C}}$ (6.14). Now consider the diagram of analytic manifolds (we assumed K to be neat):

$$\begin{array}{ccccc} P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) & \longleftarrow & \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) & \xrightarrow{p} & h(\mathbb{D}_{\mathbf{X}}) \\ \parallel & & \downarrow & & \downarrow \\ P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) & \longleftarrow & P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) & \longrightarrow & M^{\vee}(\mathbf{X})(\mathbb{C}) \end{array}$$

Let $U \cong M(K\mathbf{X})(\mathbb{C})$ be the complement of $D_{\mathbb{C}}$. The diagram shows that $\Xi^*E_{\mathbb{C}}|_U$ is canonically identified with the $P_{\mathbf{X}}(\mathbb{Q})$ -invariant sections of the pullback $p^*(E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{X}})})$ to the standard local system $\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K$. But this pullback carries the pullback of the Hermitian metric h_E . For the induced $P_{\mathbf{X}}(\mathbb{Q})$ -action on $p^*(E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{X}})})$ compatible with the usual action on the standard local system, h_E is invariant as well and descends to a smooth Hermitian metric on U .

Theorem 10.3. *Let $\frac{K}{\Delta}\mathbf{X}$ be pure p -ECSD, let (\mathcal{E}, h_E) as in 10.1 which is fully-decomposed, i.e. the unipotent radical of a stabilizer group under the action of $P_{\mathbf{X}}(\mathbb{C})$ on $M^{\vee}(\mathbf{X})(\mathbb{C})$ acts trivially on \mathcal{E} . Then the vector bundles $\Xi^*\mathcal{E}_{\mathbb{C}}$ are the unique extensions to $M(\frac{K}{\Delta}\mathbf{X})$ of these bundles on $M(K\mathbf{X})$ such that the metric Ξ^*h_E is good along the boundary divisor D_{∞} in the sense of [34, p. 242].*

Proof. Follows from the construction of the extension in [loc. cit.], cf. also [16, §3.8]. \square

Definition 10.4. *Let (\mathcal{E}, h_E) be as in 10.1. The locally free sheaf $\Xi^*\mathcal{E}$, together with the singular (along $D_{\mathbb{C}}$) Hermitian metric Ξ^*h_E is called the **Hermitian automorphic vector bundle** associated to (\mathcal{E}, h_E) on $M(\frac{K}{\Delta}\mathbf{X})$.*

Remark 10.5. *It follows directly from the construction of metrics in 10.2 that for a morphism of p -ECMSD*

$$(\alpha, \rho) : \frac{K_1}{\Delta_1}\mathbf{X}_1 \rightarrow \frac{K_2}{\Delta_2}\mathbf{X}_2$$

and given (\mathcal{E}, h_E) , with \mathcal{E} a locally free sheaf on $[M^{\vee}(\mathbf{X}_1)/P_{\mathbf{X}_1}]$ and $h_E = \{h_{E,\sigma}\}$ a collection of invariant Hermitian metrics, we have a canonical isomorphism

$$M(\alpha, \rho)^*(\Xi^*\mathcal{E}, \Xi^*h_E) \cong (\Xi^*M^{\vee}(\alpha)^*\mathcal{E}, \Xi^*M^{\vee}(\alpha)^*h_E),$$

⁸This is a slight abuse of notation, since the construction seems to depend on the particular presentation of the target stack as a quotient of M^{\vee}

in particular these isomorphisms are compatible with composition.
For the metrics this follows from the 2-isomorphism

$$(\Xi({}^K\mathbf{X}) \times_{\mathrm{spec}(\mathcal{O}_{\mathbf{X}})} \mathrm{spec}(\mathbb{C}))^{an} \rightarrow \Xi_{\mathbb{C}}({}^K\mathbf{X}),$$

given by 6.14.

Similarly, in the case of a boundary component

$$(\iota, \rho) : {}^{K_1}_{\Delta_1}\mathbf{B} \Longrightarrow {}^K_{\Delta}\mathbf{X},$$

and, say, the case of $\mathcal{O}_{\mathbf{X}} = \mathbb{Z}_{(p)}$ for simplicity, if h_E is the restriction of a $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$ -invariant metric $h_{E,1}$ on $\mathbb{D}_{\mathbf{B}}$ to $\mathbb{D}_{\mathbf{B}} \Rightarrow \mathbf{X} \subset \mathbb{D}_{\mathbf{B}}$, we have on the formal completion a canonical isomorphism

$$M(\iota, \rho)^*(\Xi^*\mathcal{E}, \Xi^*h_E) \cong (\Xi^*\mathcal{E}|_{M^{\vee}(\mathbf{B})}, \Xi^*h_{E,1})$$

taking into account that the formal isomorphism converges over \mathbb{C} . However, in general, the metrics h_E do not come from $h_{E,1}$'s as above (they are not even defined everywhere on $\mathbb{D}_{\mathbf{B}}$), but of course an Ξ^*h_E is always defined in a neighborhood of $D_{\mathbb{C}}$ on $M({}^{K_1}_{\Delta_1}\mathbf{B})(\mathbb{C})$ and the above statement makes sense.

10.6. In the reminder of this section we will describe the necessary tools from Arakelov theory that will be needed to define uniquely determined heights on (resp. arithmetic volumes of) Shimura varieties w.r.t. automorphic line bundles. We will be as minimalist as possible, stating only the properties of the objects involved that are needed in the sequel and which uniquely characterize the resulting values. We assume familiarity with classical Arakelov theory as presented in [38].

10.7. We consider the category (X, D) , where X is a smooth complex algebraic variety equipped with a divisor of normal crossings D . Morphisms $\alpha : (X, D_X) \rightarrow (Y, D_Y)$ are morphisms $\alpha : X \rightarrow Y$ which satisfy $\alpha^{-1}D_Y \subseteq D_X$. We consider an association with maps a pair (X, D) to a subspace of differential forms

$$A^{i,j}(X, D) \subset A^{i,j}(X - D),$$

for every i, j , stable w.r.t. the differentials d and d^c . In addition, for any vector bundle E on X , we consider a class \mathcal{M} of Hermitian metrics on $E|_{X-D}$, such that

- (F1) For any morphism $f : (X, D) \rightarrow (Y, D')$, f^* restricts to a map of $A^{i,j}$'s and f^* of a vector bundle with metric in \mathcal{M} is again a vector bundle with metric in \mathcal{M} .
- (F2) Each $\xi \in A^{i,j}(X, D)$ defines a current on X , and we have $d[\xi] = [d\xi]$, resp. $d^c[\xi] = [d^c\xi]$.
- (F3) The Chern character form of any vector bundle with metric in \mathcal{M} is in $\bigoplus_p A^{p,p}(X, D)$.

(F4) There is a unique way to associate with each (X, D) and exact sequence

$$E_\bullet : 0 \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow 0$$

of vector bundles on X and metrics h_{-1}, h_0, h_1 in \mathcal{M} a Bott-Chern character $\text{ch}(E_\bullet, h_{-1}, h_0, h_1) \in \bigoplus_p A^{p,p}(X, D)/(\text{im } d + \text{im } d^c)$ such that

- (a) $d d^c \text{ch}(E_\bullet, h_{-1}, h_0, h_1) = \sum_i (-1)^i \text{ch}(E_i, h_i)$,
- (b) the formation is compatible with pullback,
- (c) $\text{ch}(E_\bullet, h_{-1}, h_0, h_1) = 0$ if E_\bullet splits and $h_0 = h_{-1} \oplus h_1$,
- (d) for an exact square consisting of vector bundles $E_{i,j}$, $i, j = -1, \dots, 1$ equipped with metrics $h_{i,j}$ in \mathcal{M} , we have

$$\sum_i (-1)^i \text{ch}(E_{i,\bullet}, h_{i,-1}, h_{i,0}, h_{i,1}) = \sum_j (-1)^j \text{ch}(E_{\bullet,j}, h_{-1,j}, h_{0,j}, h_{1,j}).$$

Often convenient is the following property, which we will not need, however:

(F5) If Y is a cycle of codimension p which intersects D properly then for any Greens function g of logarithmic type along Y [38, 2.1, Definition 3], and for all $\xi \in A^{p,p}(X, D)$, $g\xi$ is locally integrable on X .

Note that by the properties (F4,a-c) the Bott-Chern characters involving only smooth metrics have to be necessarily the same as those considered in [38, 3.1, Theorem 2]. We will write also $\text{ch}(E, h_{-1}, h_0)$ for $\text{ch}(E_\bullet, h_{-1}, h_0, 0)$ where E_\bullet is the sequence $0 \rightarrow E \rightarrow E \rightarrow 0 \rightarrow 0$.

10.8. From closedness under d and d^c follows in particular that for $e_1 \in A^{p,p}(X, D)$ and $e_2 \in A^{q,q}(X, D)$ the (star-)product $e_1 * e_2 := (d d^c e_1) e_2$ is associative and commutative mod image d, d^c . If g is a form on $(X-D)(\mathbb{C})$ which can be expressed as a sum $\tilde{g} + g_0$, where \tilde{g} is a Greens form of logarithmic type [38, Definition 3, p. 43] and g is in \mathcal{M} one can define $g * h \in A^{p+q+1, p+q+1}(X-D)$ to be $\tilde{g} * \tilde{h} + g_0 d d^c \tilde{h} + h_0 d d^c \tilde{g} + h_0 d d^c g_0$, which is clearly (mod image d, d^c) independent of the way of expressing the sum.

10.9. If Y_1 and Y_2 are cycles on X of codimension p and q , respectively, *without intersection*, and \tilde{g}_1, \tilde{g}_2 be Greens forms of logarithmic type for them, we can represent their star-product (modulo image d, d^c) by

$$\tilde{g}_1 * \tilde{g}_2 = \sigma_1 \tilde{g}_1 (d d^c \tilde{g}_2) + d d^c (\sigma_2 \tilde{g}_1) \tilde{g}_2,$$

where $\sigma_1 + \sigma_2 = 1$ is a partition of unity, such that $\sigma_1 = 0$ on Y_1 and $\sigma_2 = 0$ on Y_2 . If g_1 and g_2 are the sums of the previous Greens forms with forms $g_{1,0} \in A^{p,p}(X, D)$, resp. $g_{2,0} \in A^{q,q}(X, D)$, for the star product as defined in 10.8, we have the same formula

$$g_1 * g_2 = \sigma_1 g_1 (d d^c g_2) + d d^c (\sigma_2 g_1) g_2.$$

For this note that (e.g.)

$$\sigma_1 g_1 (d d^c g_{2,0}) + d d^c (\sigma_2 g_1) g_{2,0} = g_{2,0} d d^c g_1 + \sigma_1 g_1 (d d^c g_{2,0}) - d d^c (\sigma_1 g_1) g_{2,0}.$$

However in

$$\sigma_1 g_1 (d d^c g_{2,0}) - d d^c (\sigma_1 g_1) g_{2,0}$$

$\sigma_1 g_1$ and $g_{2,0}$ are in $A^{q,q}(X, D)$. Therefore this is contained in the image of d, d^c . Using an embedded resolution of singularities and properties (F1, F2), one can extend this to non-empty intersections. We will not need this, however.

Theorem 10.10 (Burgos, Kramer, Kühn). *A class $A^{i,j}(\cdot, \cdot)$ of differential forms and class of metrics \mathcal{M} satisfying the axioms (F1)-(F4) such that \mathcal{M} includes the natural metrics $\Xi^* h_{\mathcal{E}}$ of fully decomposed automorphic vector bundles $\Xi^* \mathcal{E}$ on pure Shimura varieties exists.*

Proof. The class of differential forms is the class of log-log-forms [9, section 2.2]. Those are in particular pre-log-log forms [10, section 7.1]. The class of metrics is the class of log-singular ones [10, 4.29]. The properties above are shown respectively at the following places: (F1) [9, Proposition 2.24] for forms, [9, Proposition 4.35] for the metrics, (F2) [10, Proposition 7.6], (F3) is part of their definition of log-singular metric, (F4) [9, Proposition 4.64], (F5) (not needed here) follows from the proof of [10, Lemma 7.35]. The statement about fully decomposed automorphic vector bundles is proven in [9, Theorem 6.3]. \square

10.11. Let X be a projective regular and flat scheme over $\text{spec}(R)$, where R is any localization of \mathbb{Z} . Let D be a divisor on X such that $D_{\mathbb{C}}$ is a divisor of normal crossings. From now on we assume that classes $A^{i,j}(\cdot, \cdot)$ and \mathcal{M} satisfying (F1)-(F4) have been chosen and define in analogy to [38, p. 55]:

$$\begin{aligned} A^{p,p}(X, D) &:= \{ \omega \in A^{p,p}(X(\mathbb{C}), D(\mathbb{C})) \mid \omega \text{ real}, F_{\infty}^* \omega = (-1)^p \omega \} \\ \tilde{A}^{p,p}(X, D) &:= A^{p,p}(X, D) / (\text{im } \partial + \text{im } \bar{\partial}). \end{aligned}$$

We define arithmetic Chow groups $\widehat{\text{CH}}^p(X, D)$ by the following pushout

$$\begin{array}{ccccccc} \text{CH}^{p-1,p}(X) & \longrightarrow & \tilde{A}^{p-1,p-1}(X, D) & \xrightarrow{a} & \widehat{\text{CH}}^p(X, D) & \longrightarrow & \text{CH}^p(X) \longrightarrow 0 \\ \parallel & & \uparrow & & \uparrow & & \parallel \\ \text{CH}^{p-1,p}(X) & \longrightarrow & \tilde{A}^{p-1,p-1}(X) & \xrightarrow{a_{GS}} & \widehat{\text{CH}}_{GS}^p(X) & \longrightarrow & \text{CH}^p(X) \longrightarrow 0 \end{array}$$

where an index GS means the corresponding object as defined in [38].

We define a product

$$\widehat{\text{CH}}^p(X, D) \times \widehat{\text{CH}}^q(X, D) \rightarrow \widehat{\text{CH}}_{\mathbb{Q}}^{p+q}(X, D)$$

by defining: $a(b_1)a(b_2)$ to be $a(b_1 d d^c b_2)$ and $a(b)\hat{z}$ to be $a(b\omega(\hat{z}))$ (for the map ω of [loc. cit.]). This is well defined because the formulas hold for a_{GS} . This product is obviously associative and commutative. We also form the direct sum $\widehat{\text{CH}}^{\bullet}(X, D)_{\mathbb{Q}} := \bigoplus_p \widehat{\text{CH}}^p(X, D)_{\mathbb{Q}}$, which yields a contravariant functor (w.r.t. pullback) from the category of pairs (X, D) , where X is projective, regular, and flat over $\text{spec}(R)$ and D is a divisor on X such that $D_{\mathbb{C}}$ has only normal crossings, to the category of graded \mathbb{Q} -algebras.

10.12. If X is projective over $\text{spec}(R)$ of relative dimension d , where R is any localization of \mathbb{Z} , we have a push forward

$$\pi_* : \widehat{\text{CH}}^p(X, D) \rightarrow \widehat{\text{CH}}^{p-d}(\text{spec}(R)),$$

which on the image of a is defined by integration (see F2).

Definition 10.13. For any vector bundle \mathcal{E} with metric h in the class we define an arithmetic Chern character by

$$\widehat{\text{ch}}(\mathcal{E}, h) := \widehat{\text{ch}}_{GS}(\mathcal{E}, \tilde{h}) + a(\text{ch}(E, h, \tilde{h})),$$

where \tilde{h} is any smooth metric on $E = \mathcal{E}_{\mathbb{C}}$. In particular the graded components define Chern classes $\widehat{\text{c}}_i(\mathcal{E}, h)$ of every degree.

Lemma 10.14. This is well defined.

Proof. We have the property $\widehat{\text{ch}}_{GS}(E, \tilde{h}_2) - \widehat{\text{ch}}_{GS}(E, \tilde{h}_1) = a(\text{ch}(E, \tilde{h}_2, \tilde{h}_1))$ [38, Proposition 1, p. 86]. From properties (F4,c,d) follows that $\text{ch}(E, \tilde{h}_2, \tilde{h}_1) = \text{ch}(E, \tilde{h}_2, h) - \text{ch}(E, \tilde{h}_1, h)$. \square

The formation of Chern character (hence Chern classes) is compatible with pullback because this is true for $\widehat{\text{ch}}_{GS}(\mathcal{E}, \tilde{h})$.

10.15. For R a localization of \mathbb{Z} we have a map

$$\widehat{\text{deg}} : \widehat{\text{CH}}^1(\text{spec}(R))_{\mathbb{Q}} \rightarrow \mathbb{R}_R,$$

where \mathbb{R}_R is \mathbb{R} modulo rational multiples of $\log(p)$ for all p with $p^{-1} \in R$. We denote $\mathbb{R}_{\mathbb{Z}_{(p)}}$ also by $\mathbb{R}^{(p)}$ and $\mathbb{R}_{\mathbb{Z}[1/N]}$ by \mathbb{R}_N .

Obviously \mathbb{R}_N injects into the fibre product of all $\mathbb{R}^{(p)}$ $p \nmid N$ over $\mathbb{R}_{\mathbb{Q}}$.

Definition 10.16. Let X be as in 10.11. If $\overline{\mathcal{E}} = (\mathcal{E}, h)$ is a line bundle with metric in \mathcal{M} and Y is a cycle on X such that $Y_{\mathbb{Q}}$ intersects $D_{\mathbb{Q}}$ properly, we define the **height** with respect to $\overline{\mathcal{E}}$ of Y as follows: Choose a Greens form of logarithmic type for $Y_{\mathbb{C}}$ (exists by [38, Theorem 3, p.44]) and a smooth metric \tilde{h} on \mathcal{E} .

$$\begin{aligned} \text{ht}_{\overline{\mathcal{E}}}(Y) &:= \widehat{\text{deg}} \pi_*((Y, g) \cdot \widehat{\text{c}}_1(E, \tilde{h})^{n+1-d}) - \frac{1}{2} \int_X g \text{c}_1(E, \tilde{h})^{n+1-d} + \frac{1}{2} \int_Y \phi \\ &= \text{ht}_{(\mathcal{E}, \tilde{h}), GS}(Y) + \frac{1}{2} \int_Y \phi, \end{aligned}$$

where $\phi \in \tilde{A}^{n-d, n-d}(X, D)$ is such that $a(\phi) = \widehat{\text{c}}_1(E, h)^{n+1-d} - \widehat{\text{c}}_1(E, \tilde{h})^{n+1-d}$. The integral over Y exists because of property (F2) (consider a desingularization of Y). Here it is crucial that Y and D intersect properly.

Lemma 10.17. This is well defined.

Proof. The first line is independent of g by means of the definition of star product [38, Definition 4, p.50]. We have to see that it is independent of the choice of smooth metric \tilde{h} . Let \tilde{h}_i , $i = 1, 2$ be two different ones. Let ϕ_i be such that $a(\phi_i) = \widehat{c}_1(\mathcal{E}, h)^{n+1-d} - \widehat{c}_1(\mathcal{E}, \tilde{h}_i)^{n+1-d}$. Note that $\phi_2 - \phi_1$ is smooth. We have

$$\begin{aligned} \widehat{\deg} \pi_* \left((Y, g) \cdot \widehat{c}_1(\mathcal{E}, \tilde{h}_2)^{n+1-d} - (Y, g) \cdot \widehat{c}_1(\mathcal{E}, \tilde{h}_1)^{n+1-d} \right) &= \frac{1}{2} \int_{X(\mathbb{C})} (\phi_1 - \phi_2) d d^c g \\ &- \frac{1}{2} \int_{X(\mathbb{C})} g c_1(E, \tilde{h}_2)^{n+1-d} + \frac{1}{2} \int_{X(\mathbb{C})} g c_1(E, \tilde{h}_1)^{n+1-d} = \frac{1}{2} \int_{X(\mathbb{C})} g d d^c (\phi_2 - \phi_1) \\ &\frac{1}{2} \int_{Y(\mathbb{C})} \phi_2 - \frac{1}{2} \int_{Y(\mathbb{C})} \phi_1 = \frac{1}{2} \int_{X(\mathbb{C})} (\phi_2 - \phi_1) d d^c g - \frac{1}{2} \int_{X(\mathbb{C})} g d d^c (\phi_2 - \phi_1) \end{aligned}$$

so the sum over these differences is zero. \square

From (F1) and (F2) together with (F5) the following follows formally — using an embedded resolution of singularities (see the proof of [10, Theorem 7.33] with $\sigma_{YZ} = 0$):

$$\int_X g d d^c g_2 = \int_Y g_2 + \int_X (d d^c g) g_2$$

for any form g_2 in our class. Hence, if (F5) holds, we could simply define

$$\text{ht}_{\overline{\mathcal{E}}}(Y) := \widehat{\deg}(\pi_*((Y, g) \cdot \widehat{c}_1(\mathcal{E}, h)^{n+1-d}) - \frac{1}{2} \int_{X(\mathbb{C})} g c_1(E, h)^{n+1-d}$$

which is literally the same definition as $\text{ht}_{\overline{\mathcal{E}}, GS}$.

10.18. In any case, if there is a desingularization (\tilde{Y}, \tilde{D}) of $(Y, Y \cap D)$ which is itself a pair as considered in 10.11, we get from the projection formula [38, Theorem 3 (iii), p. 64] that

$$\text{ht}_{\overline{\mathcal{E}}}(Y) = \widehat{\deg} \pi_*(f^* \widehat{c}_1(\mathcal{E}, h))^{n+1-d}$$

for the corresponding morphism $f : (\tilde{Y}, \tilde{D}) \rightarrow (X, D)$.

10.19. We will apply the constructions above to (pure) Shimura varieties now. Assume first that all occurring K 's are neat, see 10.24 for the general case. For a pure Shimura variety (considered as defined over $\mathbb{Z}_{(p)}$ as in 10.1) $M(\frac{K}{\Delta} \mathbf{X})$ with complete, smooth w.r.t. K and projective Δ , the exceptional divisor D has normal crossings (6.9) and we define $\widehat{\text{CH}}^\bullet(M(\frac{K}{\Delta} \mathbf{X}))$ as $\widehat{\text{CH}}^\bullet(M(\frac{K}{\Delta} \mathbf{X}), D)$ as defined in 10.11 w.r.t. any class of forms and metrics satisfying (F1–4), such that the metrics of fully decomposed automorphic vector bundles are in the class (exists by Theorem 10.10). We assume that $M(\frac{K}{\Delta} \mathbf{X})$ exists — this requires that Δ is sufficiently fine and that Conjecture 6.6 is true (cf. 6.9).

Definition 10.20. Let ${}^K_{\Delta}\mathbf{X}$ be pure p -ECMSD with complete, smooth w.r.t. K and projective Δ , and $\overline{\mathcal{E}} = (\mathcal{E}, h_E)$ be an invertible sheaf \mathcal{E} on $[\mathbf{M}^\vee(\mathbf{X})/P_{\mathbf{X}}]$ (stack over $\mathbb{Z}_{(p)}!$) and $h_E = \{h_{E,\sigma}\}$ a collection of invariant Hermitian metrics as in 10.1.

The **geometric volume**

$$\mathrm{vol}_E(\mathbf{M}({}^K_{\Delta}\mathbf{X}))$$

is defined as the volume of $\mathbf{M}({}^K_{\Delta}\mathbf{X})(\mathbb{C})$ with respect to the volume form $(c_1(\Xi^* \overline{E}_{\mathbb{C}}))^d$.
The **arithmetic volume** at p

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathbf{M}({}^K_{\Delta}\mathbf{X}))$$

is defined as

$$\pi_*(\widehat{c}_1(\Xi^* \mathcal{E}, \Xi^* h_E)^d)$$

in $\widehat{\mathrm{CH}}^1(\mathrm{spec}(\mathbb{Z}_{(p)})) = \mathbb{R}^{(p)}$, where $\pi : \mathbf{M}({}^K_{\Delta}\mathbf{X}) \rightarrow \mathrm{spec}(\mathbb{Z}_{(p)})$ is the structural morphism

If $P_{\mathbf{X}}$ is a group scheme of type (P) over $\mathrm{spec}(\mathbb{Z}[1/N])$ and K is admissible for all $p \nmid N$, then we define the global **arithmetic volume**

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathbf{M}({}^K_{\Delta}\mathbf{X}))$$

in \mathbb{R}_N as the value determined by the arithmetic volumes at p (compare 10.15, see also 10.25).

If $(\alpha, \rho) : {}^{K'}_{\Delta'}\mathbf{Y}_{\mathbb{Q}} \hookrightarrow {}^K_{\Delta}\mathbf{X}_{\mathbb{Q}}$ is an embedding of rational pure Shimura data, we define the **height** at p :

$$\mathrm{ht}_{\overline{\mathcal{E}},p}(\mathbf{M}({}^{K'}_{\Delta'}\mathbf{Y})) = \mathrm{ht}_{\Xi^* \overline{\mathcal{E}}}(\overline{\mathbf{M}(\alpha, \rho)(\mathbf{M}({}^{K'}_{\Delta'}\mathbf{Y}_{\mathbb{Q}}))}^{\mathrm{Zar}}),$$

where $\mathrm{ht}_{\Xi^* \overline{\mathcal{E}}}$ is the height (10.16). Similarly for the (global) height.

Question 10.21. We mention that there is a **proportionality principle** [34] in the geometric case. This means that all polynomials of degree $\dim(\mathbf{M}({}^K_{\Delta}\mathbf{X})_{\mathbb{C}})$ in the Chern classes of an automorphic vector bundle $(\Xi^* \mathcal{E})_{\mathbb{C}}$, considered as a number, are proportional to the same expression in the Chern classes of the original bundle \mathcal{E} , computed on $\mathbf{M}^\vee(\mathbf{X})$.

Is there an analogue in the arithmetic case? — cf. also [21].

Remark 10.22. If (α, ρ) was in fact an embedding of p -integral data: ${}^{K'}_{\Delta'}\mathbf{Y} \hookrightarrow {}^K_{\Delta}\mathbf{X}$, we have, by the projection formula (cf. 10.18) and 10.5:

$$\mathrm{ht}_{\overline{\mathcal{E}},p}(\mathbf{M}({}^{K'}_{\Delta'}\mathbf{Y})) = \widehat{\mathrm{vol}}_{\mathbf{M}^\vee(\alpha)^* \overline{\mathcal{E}},p}(\mathbf{M}({}^{K'}_{\Delta'}\mathbf{Y})).$$

Lemma 10.23. The geometric and arithmetic volumes do not depend on the rational polyhedral cone decomposition Δ , i.e. not on the chosen toroidal compactification.

Proof. For each pair Δ_i , $i = 1, 2$, there is a common refinement Δ , cf. 3.22, and we have two projections (6.5):

$$\begin{array}{ccc} & M(\Delta^K \mathbf{X}) & \\ \swarrow & & \searrow \\ M(\Delta_1^K \mathbf{X}) & & M(\Delta_2^K \mathbf{X}) \end{array}$$

Furthermore this diagram is compatible with formation of $\Xi^* \mathcal{E}, \Xi^* h_E$ by 10.5. Its Chern forms are therefore transported into each other by pullback along the arrows in the diagram. Since the forms are integrable on every $M(\Delta^K \mathbf{X})$ (see [34], cf. also 10.3), the geometric volume agrees. Now, there is pullback map

$$p^* : \widehat{\text{CH}}^\bullet(M(\Delta_i^K \mathbf{X}))_{\mathbb{Q}} \rightarrow \widehat{\text{CH}}^\bullet(M(\Delta^K \mathbf{X}))_{\mathbb{Q}}$$

as well, which is a ring homomorphism and compatible with Chern classes (see (F4,b) and [38, Theorem 3, p. 84]).

Therefore the assertion for the arithmetic volume boils down to the fact that for a class $x = (z, g) \in \widehat{\text{CH}}_{GS}^{n+1}(\Delta^K \mathbf{X})$ or $x = a(g')$, where z is a zero-cycle in the fibre above p and g is a smooth top-degree Greens form and g' is a top-degree form on $M(\Delta^K \mathbf{X})(\mathbb{C})$ which is integrable on the whole $M(\Delta^K \mathbf{X})(\mathbb{C})$, we have $\widehat{\deg}(\pi_* p^*(x)) = \widehat{\deg}(\pi_* x)$, where the π are the respective structural morphisms. This is true because the push-forwards of g, g' are computed by an integral. \square

Remark 10.24. *If K is not neat, M is only a Deligne-Mumford stack. Instead of extending the theory of [9, 10] to stacks like e.g. in [13], we define the arithmetic volume in this case as follows: We take some neat admissible $K' \subset K$ and define*

$$\widehat{\text{vol}}_{\bar{\varepsilon}, p}(M(\Delta^K \mathbf{X})) := \frac{1}{[K : K']} \widehat{\text{vol}}_{\bar{\varepsilon}, p}(M(\Delta^{K'} \mathbf{X}))$$

as value in $\mathbb{R}^{(p)}$. Similarly for the global arithmetic volume in \mathbb{R}_N , assuming that both K and K' are admissible for all $p \nmid N$. It follows from 10.5, the definition of pullback, and the same reasoning as in Lemma 10.23. that this is independent of the choice of K' . In particular, if K is neat — or using any reasonable extension of Arakelov theory to stacks — this definition agrees with the previous one.

In this sense the geometric and arithmetic volume do not depend essentially on K , as they are both multiplied by the index, if K is changed. For the arithmetic case this is of course only true for admissible K , and the more primes are considered the less admissible groups there are.

Remark 10.25. *Since the canonical models can be defined as the normalization of a Zariski closure in a variety defined over some $\mathbb{Z}[1/N]$ (compactification of the moduli space of Abelian varieties), it follows that there is even a model of $M(\Delta^K \mathbf{X})$ defined over some $\mathcal{O}_{\mathbf{X}}[1/N]$, yielding the canonical ones over the various*

reflex rings $\mathcal{O}_{\mathbf{X},(p)}$ for $p|N$, such that the arithmetic volume of this model is the value determined above. This justifies a priori that our value lies actually in \mathbb{R}_N . Otherwise we will not care about this global model.

11. The arithmetic of Borcherds products

11.1. Let L be a f.d. \mathbb{Q} -vector space with non-degenerate quadratic form $Q_L \in \text{Sym}^2(L^*)$ and M be any f.d. \mathbb{Q} -vector space. Denote $\mathfrak{M} = M^* \oplus M$ with standard symplectic form. Consider the following elements of $\text{Sp}(\mathfrak{M})$:

$$\begin{aligned} g_l(\alpha) &:= \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} \\ u(\beta) &:= \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \\ d(\gamma) &:= \begin{pmatrix} 0 & -{}^t\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \end{aligned}$$

for $\alpha \in \text{Aut}(M)$, $\beta \in \text{Hom}(M^*, M)$ and $\gamma \in \text{Iso}(M, M^*)$. We denote the image of g_l and u by G_l and U respectively, and their product by P , which is a maximal parabolic of $\text{Sp}(\mathfrak{M})$.

Let R be one of $\mathbb{R}, \mathbb{Q}_p, \mathbb{A}^{(\infty)}$ or \mathbb{A} and χ its standard additive character (depends on a choice of $i \in \mathbb{C}$). Let $\text{Sp}'(\mathfrak{M}_R)$ be $\text{Sp}(\mathfrak{M}_R)$ if m is even and $\text{Mp}(\mathfrak{M}_R)$ (metaplectic double cover) if m is odd. The elements above have lifts to $\text{Sp}'(\mathfrak{M}_R)$, denoted $r''(\dots)$ such that the **Weil representation** of $\text{Sp}'(\mathfrak{M}_R)$ on $S((M^* \otimes L)_R)$ (Schwartz functions) is given by:

$$\begin{aligned} \omega(r''(g_l(\alpha)))\varphi : x^* &\mapsto \tilde{\Upsilon}(\gamma_0\alpha)\tilde{\Upsilon}(\gamma_0)^{-1}|\alpha|^{\frac{m}{2}}\varphi({}^t\alpha x^*) \\ \omega(r''(u(\beta)))\varphi : x^* &\mapsto \chi((x^*)^!Q_L \cdot \beta)\varphi(x^*) \\ \omega(r''(d(\gamma)))\varphi : x^* &\mapsto \tilde{\Upsilon}(\gamma)|\gamma|^{-\frac{m}{2}} \int_{M \otimes L^*} \varphi({}^t\gamma x)\chi(-\langle x^*, x \rangle) dx. \end{aligned}$$

$\tilde{\Upsilon}$ is a certain eighth root of unity. For $n = 1$ we have just $\tilde{\Upsilon}(\gamma) = \Upsilon(\gamma \otimes Q_L)$, where Υ is the Weil index [41]. In the third formula $|\gamma|$ is computed w.r.t. the measure dx , and so it is independent of this measure.

If a global lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ is given, with $Q_L \in \text{Sym}^2(L_{\mathbb{Z}}^*)$ and $M_{\mathbb{Z}} \subset M_{\mathbb{Q}}$, as in Borcherds' paper, we often restrict to the subspace of $S((M^* \otimes L)_{\mathbb{A}^{(\infty)}})$ given by characteristic functions of cosets in $(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*$. We denote the space — as in [3] — by $\mathbb{C}[(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*]$.

11.2. In the $n = 1$ case ($M = \mathbb{Z}$) we denote it also by $\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$, and we consider it as a representation $\text{Sp}'_2(\mathbb{Z})$, $\text{Sp}'_2(\widehat{\mathbb{Z}})$ or a suitable $\text{Sp}'_2(\mathbb{Z}/N\mathbb{Z})$. $\text{Sp}'_2(\mathbb{Z})$ is generated by 2 elements S and T , satisfying $Z = S^2 = (ST)^3$ and ($Z^4 = 1$ or $Z^2 = 1$ according to whether m is odd or even), mapping down to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, respectively, in $\mathrm{Sp}_2(\mathbb{Z})$ and acting explicitly as:

$$\begin{aligned} T\chi_\kappa &= \exp(2\pi i Q_L(\kappa))\chi_\kappa, \\ S\chi_\kappa &= \frac{\Upsilon_\infty(Q_L)^{-1}}{\sqrt{D}} \sum_{\delta \in L_{\mathbb{Z}}^*/L_{\mathbb{Z}}} \exp(-2\pi i \langle \delta, \kappa \rangle) \chi_\delta. \end{aligned}$$

Note that the occurring correction factor $\tilde{\Upsilon}_f(1) = \Upsilon_f(Q_L)$ is the product over the local $\Upsilon_p(Q_L)$'s which is, however, the same as $\Upsilon_\infty(Q_L)^{-1}$ and we have $\Upsilon_\infty(Q_L) = \exp(2\pi i(p - q))$, if the signature of Q is (p, q) . These are the formulas used in [3]. We will need the following facts about the Weil representation, too: The Weil representation of $\mathrm{Sp}'_2(\widehat{\mathbb{Z}})$, or $\mathrm{Sp}'_2(\mathbb{Z})$, always factors through $\mathrm{Sp}'_2(\mathbb{Z}/N\mathbb{Z})$, where N is related to the exponent of $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$. If m is even, the Borel subgroup $\begin{pmatrix} * & * \\ & * \end{pmatrix} \subset \mathrm{Sp}'_2(\mathbb{Z}/N\mathbb{Z})$ acts on $\chi_{L_{\widehat{\mathbb{Z}}}}$ by the character $\Upsilon(\alpha Q_L)\Upsilon(Q_L)^{-1}$.

By construction of the Weil representation, we have

$$\mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) = \bigotimes_{p|D} \mathrm{Weil}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}), \quad (10)$$

where $\mathrm{Weil}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p})$ is actually a representation of $\mathrm{Sp}'_2(\mathbb{Z}_p)$. If $L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}$ is cyclic of prime order or of order 4, we have

$$\mathrm{Weil}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) = \mathrm{Weil}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p})^+ \oplus \mathrm{Weil}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p})^-$$

(decompositions into odd and even functions), and both summands are *irreducible*. (If $p = 2$ and the order is 2, $\mathrm{Weil}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p})^- = 0$.) Together with (10) this yields the decomposition of the Weil representation of L into irreducible representations, if D is square-free, resp. $\frac{D}{2}$ square-free and $L_{\mathbb{Z}_2}^*/L_{\mathbb{Z}_2}$ cyclic of order 4.

11.3. Let $L_{\mathbb{Q}}$ be a non-degenerate quadratic space of signature (p, q) . Consider a one-dimensional space $M_{\mathbb{Q}} = \mathbb{Q}$. We canonically identify $M^* \otimes L = L$.

Let N be a maximal negative definite subspace of $L_{\mathbb{R}}$. We have the Gaussian $\varphi_{\infty}^0 \in S(L_{\mathbb{R}}) \otimes \mathcal{C}(\mathbb{D}_{\mathbf{O}})$, defined by

$$\varphi_{\infty}^0(x, N) := e^{-\pi \langle x, \alpha_N x \rangle},$$

where $\alpha_N \in \mathrm{End}(M)$ acts as -1 on N and as +1 on the orthogonal complement. It satisfies

$$\varphi_{\infty}^0(hx, hN) = \varphi_{\infty}^0(x, N) \quad \forall h \in \mathrm{SO}(L_{\mathbb{R}}).$$

For any $\varphi \in S(L_{\mathbb{A}(\infty)})$, define the theta function

$$\Theta(g', N, h; \varphi) = \sum_{v \in L_{\mathbb{Q}}} \sum \omega(g')(\varphi_{\infty}(\cdot, N) \otimes \omega(h)\varphi)(x)$$

as a function of $g' \in \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}})$, $N \in \mathbb{D}_{\mathbf{O}}$, $h \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$. Here ω is the Weil representation (11.1).

We have (cf. [40]) for any $\gamma \in \mathrm{SO}(L_{\mathbb{Q}})$ and $\gamma' \in \mathrm{Sp}'(\mathfrak{M}_{\mathbb{Q}})$ (canonical lift of $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})$ to $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}})$),

$$\Theta(\gamma'g', \gamma N, \gamma h; \varphi) = \Theta(g', N, h; \varphi)$$

and for $g'_1 \in \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}})$ and $h_1 \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$:

$$\Theta(g'g'_1, N, hh_1; \varphi) = \Theta(g', N, h; \omega(g'_1)\omega(h_1)\varphi).$$

By these invariance properties, we may consider Θ as a function

$$\mathrm{Sp}'(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}}) \times \mathrm{M}({}^K\mathbf{O}(L))(\mathbb{C}) \rightarrow (S(L_{\mathbb{A}^{(\infty)}})^K)^*.$$

We have (cf. [22, p. 10]):

$$\omega(k'_\infty)\varphi_\infty^0(\cdot, N) = \chi_l(k'_\infty)\varphi_\infty^0(\cdot, N)$$

for $l = \frac{p-q}{2}$ and $k_\infty \in K'_\infty$ (inverse image of $K_\infty = \mathrm{SO}(2)$ in $\mathrm{Sp}'_2(\mathbb{R})$). It follows that

$$\Theta(g'k'_\infty k', N, h) = \chi_l(k'_\infty)({}^t\omega(k'))^{-1}\Theta(g', N, h)$$

for all $k'_\infty \in K'_\infty$ and $k' \in K'$ (inverse image of $\mathrm{Sp}_2(\widehat{\mathbb{Z}})$ in $\mathrm{Sp}'_2(\mathbb{A}^{(\infty)})$). Now consider an automorphic function

$$F : \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}}) \rightarrow S(L_{\mathbb{A}^{(\infty)}}),$$

satisfying

$$F(g'k'_\infty k') = \chi_{-l}(k'_\infty)\omega(k')^{-1}F(g')$$

and invariant under some compact open subgroup $K \subset \mathrm{SO}(\mathbb{A}^{(\infty)})$.

Define the **Borcherds lift** of F as

$$\Phi(N, h; F) = \int_{\mathrm{Sp}'_2(\mathbb{Q}) \backslash \mathrm{Sp}'_2(\mathbb{A})}^{\bullet} \Theta(g', N, h; F(g')) \, dg',$$

where \bullet denotes Borcherds' regularization [3], described in [22] in the adelic language. dg' is the Tamagawa measure. Note that this formation does not depend on K . By definition, we have

$$\Phi(N, h\rho; F) = \Phi(N, h; \omega(\rho)F). \quad (11)$$

It compares to the language of Borcherds as follows: If $K \subset \mathrm{SO}(\mathbb{A}^{(\infty)})$ is the discriminant kernel with respect to a lattice $L_{\mathbb{Z}}$, we may consider Θ as valued in $\mathbb{C}[(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})]^*$ and F , valued in $\mathbb{C}[(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})]$, relates to a usual modular form (if l is odd, of half integer weight) by

$$y^{l/2}F(g'_\tau) = f(\tau) \quad \tau = x + iy \in \mathbb{H}.$$

Lemma 11.4. *We have*

$$\Phi(\iota(N), 1; F) = \Phi_{\text{classic}}(\iota, L_{\mathbb{Z}}, N, f)$$

for $N \in \text{Grass}^-(L_{\mathbb{R}})$ and according to $\iota : \text{Grass}^-(L_{\mathbb{R}}) \hookrightarrow \mathbb{D}_{\mathbf{O}}$. ι is a parameter in Borcherds' theory (cf. [3, p. 46]).

Proof. In 11.2, we saw that the Weil representation composed with the inclusion $\text{Sp}'_2(\mathbb{Z}) \rightarrow \text{Sp}'_2(\widehat{\mathbb{Z}})$ acts on the subspace $\mathbb{C}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}] \subset S(L_{\mathbb{A}(\infty)})$ by the same formulas used in [3]. \square

We call f , resp. F , nearly holomorphic, if F is holomorphic on \mathbb{H} and has a Fourier expansion

$$f(\tau) = \sum_{k \in \mathbb{Q}} a_k \exp(2\pi i k \tau)$$

with $a_k = 0$ unless $k \gg -\infty$.

11.5. We recall some definitions from [17]. Let $L_{\mathbb{Z}}$ be a lattice of dimension m with non-degenerate quadratic form in $\text{Sym}^2(L_{\mathbb{Z}}^*)$. Let $M_{\mathbb{Z}}$ be a lattice of dimension n with quadratic form, not necessarily integer valued. We assume $m - n \geq 1$ throughout. (It suffices even to have $\widehat{\mathbb{Z}}$ -lattices.) Let κ be a coset in $(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*$.

We defined the following Euler products [17, 10.2, 10.3]:

$$\begin{aligned} \lambda(L; s) &:= \prod_{\nu} \lambda_{\nu}(L; s), \\ \mu(L, M, \kappa; s) &:= \prod_{\nu} \mu_{\nu}(L, M, \kappa; s), \end{aligned}$$

where [17, Definition 5.9]:

$$\begin{aligned} \lambda_p(L_{\mathbb{Z}}; s) &:= \frac{\text{vol}(\text{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s))}{\prod_{i=1}^s (1 - p^{-2i})}, \\ \mu_p(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) &:= \frac{\text{vol}(\text{I}(M, L \oplus H_{\mathbb{Z}_p}^s)(\mathbb{Q}_p) \cap \kappa \oplus H_{\mathbb{Z}_p}^s \otimes M_{\mathbb{Z}_p}^*)}{|d(L_{\mathbb{Z}_p})|^{\frac{n}{2}} |d(M_{\mathbb{Z}_p})|^{-s + \frac{1+n-m}{2}}} \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s), \end{aligned}$$

and [17, Definition 6.1]:

$$\begin{aligned} \lambda_{\infty}(L_{\mathbb{Z}}) &:= \Gamma_{m-1, m}(s) \\ \mu_{\infty}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa) &:= \Gamma_{n, m}(s). \end{aligned}$$

The μ functions are related to Whittaker integrals occurring as Euler factors of Eisenstein series. The coefficient μ_p is equal to the corresponding p -factor ([17, Theorem 4.5]), whereas the infinite factor is only indirectly related to μ_{∞} ([17, Theorem 4.7]). In the case $n = 1$ it occurs as a limit of the Archimedean factor up to a factor $-|2d(M_{\mathbb{Z}})|^{s-s_0}$, where $d(M_{\mathbb{Z}})$ is the discriminant of the associated

bilinear form (a number in this case) on $M_{\mathbb{Z}}$. It is therefore convenient to define [17, Section 10]:

$$\tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) = |2d(M_{\mathbb{Z}})|_{\infty}^{-\frac{1}{2}s} \mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s).$$

because the derivative in s of this quantity is actually related to the integral of a Borcherds form (cf. Theorem 11.14).

We define also:

$$\tilde{\lambda}(L; s) := |d(L_{\mathbb{Z}})|_{\infty}^{\frac{1}{2}s} \lambda(L_{\mathbb{Z}}; s).$$

The functions μ_p and λ_p can be computed explicitly. See [17, Theorem 8.1, 8.2]. For explicit computations of $\tilde{\mu}$ and $\tilde{\lambda}$ from this, see [17, Example 11.10–11.15]. We will continue the discussion of these functions in 14.1.

11.6. Assume from now on that the signature of L is $(m-2, 2)$. Borcherds proved in [3, Theorem 13.3] that $\Phi(F)$ is roughly the logarithm of the Hermitian norm of a modular form on $M(\frac{K}{\Delta}\mathbf{O}(L))$, which has a striking product expansion. The task of the rest of this section is a translation of this expansion into the adelic language.

Definition 11.7. Let \mathcal{E} be the restriction of the tautological line bundle on $\mathbb{P}(L)$ to the zero-quadric $M^{\vee}(\mathbf{O})$. It carries a $\mathrm{SO}(L)$ action. $E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{O}})}$ carries a natural Hermitian metric h_E , too, which we normalize as follows:

$$h_E(v) := -\frac{1}{2}e^{-C}\langle v, \bar{v} \rangle,$$

where $C = \gamma + \log(2\pi)$ and $\gamma = -\Gamma'(1)$ is Euler's constant. Denote $\overline{\mathcal{E}} = (\mathcal{E}, h_E)$.

11.8. Let $I_{\mathbb{Q}}$ be an isotropic line in $L_{\mathbb{Q}}$ and $z' \in L_{\mathbb{Q}}$ a vector with $\langle z', I \rangle \neq 0$. We fix a measure dx on $\mathbb{A}^{(\infty)}$ which has the property, that the measure of the set

$$\{\langle \delta, z' \rangle \mid \delta \in I_{\mathbb{A}^{(\infty)}}, a_k(x + \delta) = a_k(x) \text{ for all } k \in \mathbb{Q}\}$$

is an integer for all $x \in L_{\mathbb{A}^{(\infty)}}$. Since f is a finite linear combination of input modular forms considered by Borcherds, whose Fourier coefficients lies in the subspace $\mathbb{C}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$ for certain lattices $L_{\mathbb{Z}}$, this requirement does indeed make sense. We let $\chi_{\alpha} : \mathbb{A}^{(\infty)} \rightarrow \mathbb{C}^*$, $\alpha \in \mathrm{Iso}(\mathbb{Z}, \mathbb{Z}(1))$, be the corresponding additive character, i.e. such that $d\delta$ is self-dual with respect to it, and such that $\chi_{\alpha}|_{\mathbb{Q}} = \exp(\alpha(q \cdot))$ for a $q > 0$.

We let $U := (I^{\perp}/I) \otimes I \subseteq \Lambda^2 L$. It acts (Lie algebra action, see proof of Proposition 5.5) on L , and we have a (rational) boundary morphism

$$\mathbf{H}_0[U_{\mathbb{Q}}, 0] \implies \mathbf{O}(L_{\mathbb{Q}}).$$

We may identify U^* with $(I^{\perp}/I) \otimes I^*$ by means of Q_L . z' induces the following quadratic form on U :

$$Q_{z'}(k \wedge z_1) = Q_L((k \wedge z_1) \circ z') = (\langle z_1, z' \rangle)^2 Q_L(k).$$

Here \circ denotes the Lie algebra action of U on L . Its inverse on U^* is:

$$Q_{z'}(k \wedge z'_1) = \left(\frac{\langle z, z'_1 \rangle}{\langle z, z' \rangle} \right)^2 Q_L(k)$$

for an arbitrary $z \in I$.

We may also embed U^* in I^\perp by associating

$$\lambda = k \wedge z'_1 \mapsto \lambda_{z'} := \frac{\langle z, z'_1 \rangle}{\langle z, z' \rangle} \tilde{k}, \quad (12)$$

where $\tilde{k} \in I^\perp$ is the unique lift of k with $\langle z', \tilde{k} \rangle = 0$. The embedding may be seen as the composition

$$U^* \xrightarrow{\langle \cdot, \cdot \rangle_{z'}} U \xrightarrow{\circ z'} \langle I, z' \rangle^\perp.$$

The pullback of Q_L is the quadratic form $Q_{z'}$.

Definition/Lemma 11.9. *If $\sigma \subset U_{\mathbb{R}}$ is a sufficiently small cone, for example a Weyl chamber [3, §10], the Weyl vector $\rho(F, I, \sigma) \in U_{\mathbb{Q}}^*$ is characterized by*

$$8\pi\rho(F, I, \sigma)(\nu) = \sqrt{Q_{z'}(\nu)}\Phi(\langle \nu \rangle, 1, F^{U, z'}) \quad \text{for } \nu \in \sigma \subset U_{\mathbb{R}},$$

where $F^{U, z'}$ as a $S(U_{\mathbb{A}(\infty)})$ -valued function is defined as:

$$F^{U, z'}(g')(w) := C^{-1} \int_{\delta \in I_{\mathbb{A}(\infty)}} F(g')(w \circ z' + \delta) d\langle z', \delta \rangle$$

(any invariant measure). It does not depend on the choice of z' .

Proof. Compare [3, p. 40] for this characterization of the Weyl vector. First it is independent of z' : Scaling z' to $\beta z'$ changes the $w \circ z'$ to $\beta w \circ z'$. $F^{U, z'}(w, z')$ changes to $\beta^{-1} F^{U, z'}(\beta w, g')$. Since the argument of the quadratic form on $U_{\mathbb{Q}}$ and the argument in $K_{\mathbb{A}(\infty)}$ of $F^{U, z'}$ are both scaled by β , $\Psi(\dots)$ is only scaled by β^{-1} coming from scaling $F^{U, z'}$, hence invariance of the Weyl vector. (Observe that σ is invariant under scaling.) Replacing z' by $\exp(w)z'$ for $w \in U$ does neither change $F^{U, z'}$ nor $Q_{z'}$. \square

Definition/Lemma 11.10. *Let F be a modular form as above, I, z', α be chosen and $W \subset U_{\mathbb{R}}$ be a Weyl chamber. The associated adelic Borcherds product is the formal function*

$$\begin{aligned} \text{BP}(F, I, z', \alpha, W) &:= C^{-\frac{a_0(0)}{2}} \prod_{\substack{\delta \in I_{\mathbb{A}(\infty)} \\ \chi_{\alpha}(\langle z', \delta \rangle) \neq 1}} (1 - \chi_{\alpha}(\langle z', \delta \rangle))^{\frac{a_0(\delta)}{2}} d\langle z', \delta \rangle \\ &\cdot \prod_{\substack{\lambda \in U_{\mathbb{Q}}^* \\ \langle \lambda, W \rangle > 0}} \prod_{\delta \in I_{\mathbb{A}(\infty)}} (1 - \chi_{\alpha}(\langle z', \delta \rangle) [\frac{\lambda}{C}])^{a_{Q_{z'}(\lambda)}(\lambda_{z'} + \delta)} d\langle z', \delta \rangle \end{aligned}$$

in $\mathbb{C}[[U_{\mathbb{Z}}^* \cap C^\vee]][U_{\mathbb{Z}}^*]$ for a certain appropriate lattice $U_{\mathbb{Z}} \subseteq U_{\mathbb{Q}}$. Here C is the conductor, i.e. the volume of $\widehat{\mathbb{Z}}$ with respect to dx .

$\text{BP}(F, I, z', \alpha, W)$ is independent of the chosen measure.

Proof of the independence. Rewrite the first integral as

$$\left. \frac{\prod_{\delta \in I_{\mathbb{A}(\infty)}} (1 - \chi_\alpha(\langle z', \delta \rangle) [C^{-1} \lambda])^{\frac{a_0(\delta)}{2} d\langle z', \delta \rangle}}{(1 - [C^{-1} \lambda])^{\frac{a_0(0)}{2}}} \right|_{\lambda=0}.$$

Now consider the multiplicative integral

$$\prod_{\delta \in (z')^{-1}(C\widehat{\mathbb{Z}}) + a} (1 - \chi_\alpha(\langle z', \delta \rangle) [\lambda C^{-1}])^{d\langle z', \delta \rangle} = (1 - \chi_\alpha(\langle z', a \rangle) [C^{-1} \lambda]). \quad (13)$$

All multiplicative integrals occurring in $\text{BP}(F, I, z', \alpha, W)$ split into a product of these integrals by hypothesis on the measure. Now changing the measure to $N d\delta$, we get the expression

$$\begin{aligned} & \prod_{\delta \in (z')^{-1}(C\widehat{\mathbb{Z}})} (1 - \chi_\alpha(N^{-1}\langle z', a \rangle) \chi_\alpha(N^{-1}\langle z', \delta \rangle) [(NC)^{-1} \lambda])^{dN^{-1}\langle z', \delta \rangle} \\ &= \prod_{i \in \mathbb{Z}/N\mathbb{Z}} (1 - \chi_\alpha(N^{-1}\langle z', a \rangle) \zeta^i [(NC)^{-1} \lambda]) \\ &= (1 - \chi_\alpha(\langle z', a \rangle) [C^{-1} \lambda]) \end{aligned}$$

here ζ is some primitive N -th root of unity in \mathbb{C} . This shows invariance of the multiplicative integral (13). We have to check what happens to

$$\frac{1}{(1 - [\lambda C^{-1}])^{\frac{a_0(0)}{2}}}.$$

It changes to $\frac{1}{(1 - [\lambda (NC)^{-1}])^{\frac{a_0(0)}{2}}}$ hence gets multiplied by

$$(1 + [\lambda (CN)^{-1}] + [2\lambda (CN)^{-1}] + \cdots + [(N-1)\lambda (CN)^{-1}])^{\frac{a_0(0)}{2}},$$

which for $\lambda = 0$ is equal to $N^{\frac{a_0(0)}{2}}$. $C^{-\frac{a_0(0)}{2}}$ gets $(NC)^{-\frac{a_0(0)}{2}}$. Therefore the whole expression $\text{BP}(F, I, z', \alpha, W)$ is invariant. \square

We start by observing the following lemmas about the adelic Borchers products:

Lemma 11.11. $\text{BP}(F, I, z', \alpha, W)$ multiplied with the trivialization $(s_{z'})^{\otimes \frac{a_0(0)}{2}}$ (8.2) is invariant under $z' \mapsto \beta z'$ for $\beta \in \mathbb{Q}_{>0}$, i.e. we have

$$\text{BP}(F, I, \beta z', \alpha, W) = \beta^{-\frac{a_0(0)}{2}} \text{BP}(F, I, z', \alpha, W).$$

Proof. Changing z' in the various $\langle z', \delta \rangle$ has the same effect as scaling the measure, hence (by the last lemma) as multiplying C by β . The expression $\lambda_{z'}$ gets divided by β (12), $Q_{z'}(\lambda)$ gets divided by β^2 . Substituting $\beta^2 \lambda$ for λ , we get the claimed expression. \square

Lemma 11.12. *For $w = k \wedge z_1 \in U_{\mathbb{Q}}$, $\text{BP}(F, I, \exp(w)z', \alpha, W)$ is equal to $\text{BP}(F, I, z', \alpha, W)$ transformed by the homomorphism $[\lambda] \mapsto \exp(\alpha(\lambda(w)))[\lambda]$.*

Proof. Let $w = k \wedge z_1$ and

$$z'' := \exp(w)z' = z' + \langle z_1, z' \rangle k - (\langle k, z' \rangle + \langle z_1, z' \rangle Q(k))z_1.$$

First $Q_{z'} = Q_{z''}$ because $\langle z, z' \rangle = \langle z, z'' \rangle$.

We have (see (12) for the definition):

$$\lambda_{z''} + \delta = \lambda_{z'} + \delta - \frac{\lambda(w)}{\langle z_1, z' \rangle} z_1.$$

If we substitute $\delta \mapsto \delta + \frac{\lambda(w)}{\langle z_1, z' \rangle} z_1$, we get the same by applying $[\lambda C^{-1}] \mapsto \chi_{\alpha}(\lambda w)[\lambda C^{-1}]$, i.e. by $[\lambda] \mapsto \exp(\alpha(\lambda w))[\lambda]$. \square

11.13. Using the q -expansion principle 8.7, we will show the following strengthening of the main results [3, Theorem 13.3] and [22, Theorem 2.12] (resp. [7, Theorem 4.11]).

Let $p \neq 2$ be a prime⁹. Let $L_{\mathbb{Z}_{(p)}} \subset L_{\mathbb{Q}}$ be a lattice such that the quadratic form is unimodular. For this to exist, it is necessary that p does not divide a minimal discriminant of $L_{\mathbb{Q}}$.

Assume that f has Fourier coefficients of the form $a_q = \chi_{L_{\mathbb{Z}_p}} \otimes a_q^{(p)}$, $a_q^{(p)}$ as in 11.3 with values in $S(L_{\mathbb{A}^{(\infty, p)}})$. f then is invariant under an (w.l.o.g.) *admissible* compact open subgroup $K \subseteq \text{SO}(\mathbb{A}^{(\infty)})$, with Fourier coefficients $a_q \in S(L_{\mathbb{A}^{(\infty)}})$, $q \in \mathbb{Q}$.

Theorem 11.14. *Assume 6.6.*

There is an N , $p \nmid N$ such that for $R = \mathbb{Z}_{(p)}[\zeta_{N, \mathbb{C}}] \subset \mathbb{C}$, and up to multiplication of F by a large constant $\in \mathbb{Z}$, we have

1. *For any r.p.c.d. Δ , there is a section*

$$\Psi(F) \in H^0(\text{M}_{\Delta}^K(\mathbf{O})_R, (\Xi^* \mathcal{E})_{\text{horz.}}^{\otimes a_0(0)/2})$$

such that on $\text{M}^K(\mathbf{O})(\mathbb{C})$, we have (see 11.7 for the definition of h)

$$\Phi(\nu, h; F) = -2 \log h(\Psi(\nu, h; F)).$$

There is a locally constant, invertible function Λ on $\text{M}_{\Delta}^K(\mathbf{O})_R$ such that

$$\Lambda \Psi(F) \in H^0(\text{M}_{\Delta}^K(\mathbf{O}), (\Xi^* \mathcal{E})_{\text{horz.}}^{\otimes a_0(0)/2})$$

(i.e. $\Lambda \Psi(F)$ is defined over $\mathbb{Z}_{(p)}$).

⁹ $\neq 2$ in order to be able to apply the Main Theorem 6.9

2.

$$\operatorname{div}(\Psi(\nu, h, F)^2) = \sum_{q < 0} Z(L_{\mathbb{Z}}, < -q >, a_q; K) + D_{\infty}$$

(cf. 9.6), where D_{∞} has support only within the support of the exceptional divisor.

3.

$$\begin{aligned} & \int_{M(K\mathbf{O})_{\mathbb{C}}} \log \Xi^* h(\Lambda \Psi(\nu, h; F)) c_1(\Xi^* \mathcal{E}, \Xi^* h)^{m-2} \\ & \equiv \sum_{q < 0} \operatorname{vol}_{\mathcal{E}}(M(K\mathbf{O})) \cdot \left. \frac{d}{ds} \tilde{\mu}(L, < -q >_{\mathbb{Z}}, a_q; s) \right|_{s=0} \end{aligned}$$

in $\mathbb{R}^{(p)}$ (see 11.5 for the definition of $\tilde{\mu}$).

4. For any saturated isotropic line $I \subset L_{\mathbb{Z}(p)}$ and primitive isotropic vector z' in $L_{\mathbb{Z}(p)}$ with $\langle I, z' \rangle \neq 0$ consider the associated boundary morphism (5.5)

$$\iota : \mathbf{H}_0[(I^{\perp}/I) \otimes I, 0] \rightarrow \mathbf{O}(L).$$

Using the trivialization $(s_{z'})^{\otimes a_0(0)/2}$ (8.2), we get Fourier coefficients (8.5, 8.6) which are valid on a set of the form $\{\alpha\} \times (x + U_{\mathbb{R}} + \sigma) \times \{1\}$, where $\sigma \subset U_{\mathbb{R}}(-1)$ is some r.p.cone. The formal function

$$\sum_{\lambda \in U_{\mathbb{Q}}^*} a_{\iota, z', \lambda, \alpha, 1, \sigma}(\Psi(F))[\lambda]$$

is, up to multiplying $\Psi(F)$ by a constant of absolute value 1, given by:

$$[\rho(F, I, W)] \cdot \operatorname{BP}(F, I, z', \alpha, W). \quad (14)$$

Remark: The expansion in 4. suffices to know every other expansion, i.e. for $\rho \neq 1$, by property (11).

Proof. First observe that F is a finite linear combination of functions valued in $\mathbb{C}[L_{\mathbb{Z}}/L_{\mathbb{Z}}^*]$ for various lattices $L_{\mathbb{Z}}$ of $L_{\mathbb{Z}(p)}$. Hence, we may assume that F is of this kind, observing that all properties are stable under taking finite linear combinations, resp. change of K . Hence w.l.o.g. F corresponds (as above) to an f considered in [3].

Recall the normalization of the Hermitian metric h (11.7). Consider the boundary morphism as in 4. associated with some I and a $< z' >$:

$$\iota : \mathbf{H}_0[(I^{\perp}/I) \otimes I, 0] \rightarrow \mathbf{O}(L)$$

$\frac{\Psi}{s_{z'}}$ may be seen as a function on $L_{0, \mathbb{C}} = < z', I >^{\perp}$. We denote the corresponding variable by Z . We have

$$h_{\mathcal{E}}(\Psi) = \left| \frac{\Psi}{s} \right|^2 h_{\mathcal{E}}(s),$$

and

$$h_{\mathcal{E}}(s) = ((2\pi)^{-1}e^{-\gamma}\langle \Im(Z), \Im(Z) \rangle (2\pi)^2)^{\frac{a_0(0)}{2}}.$$

Hence we get

$$\log h_{\mathcal{E}}(\Psi) = \log \left| \frac{\Psi}{s} \right|^2 + \frac{a_0(0)}{2} (\log(\langle \Im(Z), \Im(Z) \rangle) + \log(2\pi) - \gamma)$$

and 1. follows from [3, Theorem 13.3], 3. is essentially [22, Theorem 2.12 (ii)], using [17, Theorem 4.7 and section 10.2]. See [7, Theorem 4.11] for an independent proof (use [17, Lemma 7.3] for the comparison). Observe that the function Ψ differs from the one in [22] or [3] by $(2\pi i)^{\frac{a_0(0)}{2}}$. The constant of the Hermitian metric has been adjusted, such that 3. holds without an additive constant. Compare also to STEP 1 of the proof of Main Theorem 14.5, where the same constant appears naturally, too.

That some multiple of F exists, such that *all* coefficients are integral and even is due to the classical q -expansion principle. Further multiplication with the order of the character of $\mathrm{SO}(\mathbb{Q})$ occurring in [3], we get a section of the *complex* bundle $\Xi^* \mathcal{E}_{\mathbb{C}}$.

It suffices to prove 4. for any choice of z' and $\alpha : 1 \mapsto 2\pi i$. For note that change of z' by β' changes BP by $\beta^{-a_0(0)/2}$ (11.11) and $s_{z'}^{\otimes a_0(0)/2}$ by $\beta^{a_0(0)/2}$. Changing z' by $\exp(w)z'$ has the effect of translating by w in the $U_{\mathbb{C}}$ -domain. Hence a formal function gets transformed by $[\lambda] \mapsto \exp(2\pi i \lambda(w))[\lambda]$. This is the same effect as observed by BP (11.12). The Weyl vector is independent of z' so the above transformation has the effect of multiplication by a constant of absolute value 1.

Choose $z = z_{\mathrm{Bor}} \in L_{\mathbb{Z}} \cap I$ and $z'_{\mathrm{Bor}} \in L_{\mathbb{Z}}^*$ as in [3], i.e. z_{Bor} primitive and isotropic and $\langle z_{\mathrm{Bor}}, z'_{\mathrm{Bor}} \rangle = 1$.

We may now show 4. for $z' := z'_{\mathrm{Bor}} - Q(z'_{\mathrm{Bor}})z_{\mathrm{Bor}}$, which is now isotropic, but satisfies still $\langle z_{\mathrm{Bor}}, z' \rangle = 1$. z' may not lie in $L_{\mathbb{Z}}^*$, however.

As measure we may choose the standard invariant measure on $\mathbb{A}^{(\infty)}$, giving $\widehat{\mathbb{Z}}$ the volume one. The trivialization now coincides (up to the factor $(2\pi i)^{\frac{a_0(0)}{2}}$ that we incorporated in the characterization of $\Psi(F)$) with the one given in [3], up to identification of I^{\perp}/I with $I \otimes (I^{\perp}/I)$ with by $k \mapsto k \otimes z_{\mathrm{Bor}}$. This interchanges Q and $Q_{z'}$.

Recall from [3, Theorem 13.3 (v)] that the Fourier expansion with respect to this particular choice is (up to a constant of absolute value 1) given by

$$\prod_{\delta \in \mathbb{Z}/N\mathbb{Z}} (1 - e(\delta/N))^{a_{\delta z/N}(0)/2} \cdot [\rho(U, W, F_U)] \cdot \prod_{\substack{\lambda \in (\langle z \rangle^{\perp} / \langle z \rangle)^* \\ \langle \lambda, W \rangle > 0}} \prod_{\substack{\delta \in (L_{\mathbb{Z}})^* / L_{\mathbb{Z}} \\ \delta|_{z^{\perp}} = \lambda}} (1 - e(\langle \delta, z' \rangle) [\lambda])^{a_{\lambda^2/2}(\delta)}. \quad (15)$$

The first product in BP may be rewritten as

$$\prod_{\delta \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}} (1 - e(\delta))^{\frac{a_0(\delta z)}{2}},$$

and coincides with the product in Borchers' expression. The Weyl vector is the same, too.

Claim: If $\langle z, z' \rangle = 1$, there is a bijection between

$$\{\lambda \in U_{\mathbb{Q}}^*, \delta \in \mathbb{Q}/\mathbb{Z} = \mathbb{A}^{(\infty)}/\widehat{\mathbb{Z}} \mid \lambda_{z'} + \delta z \in L_{\mathbb{Z}}^*\}.$$

and

$$\{k_{\text{Bor}} \in (< z >_{\mathbb{Z}}^\perp / < z >_{\mathbb{Z}})^*, \delta_{\text{Bor}} \in L_{\mathbb{Z}}^*/L_{\mathbb{Z}}, \delta_{\text{Bor}}|_{z^\perp} = k_{\text{Bor}}\}$$

Proof: The map is given sending (λ, δ) to $(k_{\text{Bor}}, \delta_{\text{Bor}}) = (\lambda z, \lambda_{z'} + \delta z)$. For $v \in < z >_{\mathbb{Z}}^\perp$, we have

$$\langle k_{\text{Bor}}, v \rangle = \langle \lambda z, v \rangle = \langle \lambda_{z'}, v \rangle = \langle \lambda_{z'} + \delta z, v \rangle \in \mathbb{Z},$$

hence $k_{\text{Bor}} \in (< z >_{\mathbb{Z}}^\perp / < z >_{\mathbb{Z}})^*$.

In the other direction, we map $(k_{\text{Bor}}, \delta_{\text{Bor}})$ to $(\lambda, \delta) = (k_{\text{Bor}} \otimes z', \langle \delta_{\text{Bor}}, z' \rangle)$. These maps are inverse to each other. One direction is clear; for the other we have to show:

$$\delta_{\text{Bor}} \equiv \widetilde{k}_{\text{Bor}} + \langle \delta_{\text{Bor}}, z' \rangle z \pmod{L_{\mathbb{Z}}},$$

where $\widetilde{k}_{\text{Bor}}$ is the representative in $< z >_{\mathbb{Q}}^\perp$ that satisfies $\langle \widetilde{k}_{\text{Bor}}, z' \rangle = 0$. This follows from the condition $\delta_{\text{Bor}}|_{z^\perp} = k_{\text{Bor}}$.

The claim shows that the second product occurring in Borchers' Fourier expansion is equal to the second product appearing in BP. This finishes the proof of 4.

We now show the integrality properties stated in 1. and 2.

We can multiply $\Psi(F)$ by an appropriate complex number of absolute value 1 on every connected component and hence may assume that *one* Fourier expansion for each connected component is given *precisely* by

$$[\rho(\omega(\rho)F, I, W)] \text{BP}(\omega(\rho)F, I, z', \alpha, W). \quad (16)$$

We get a finite number of different expansions with coefficients in some ring $R = \mathbb{Z}_{(p)}[\zeta_N, \mathbb{C}]$, $p \nmid N$ which we may assume to contain the ring described by 8.7. By multiplying F again by a large integer, we may also assume that all Weyl vectors involved are integral.

Therefore by 8.7 $\Psi(F)$ extends to an (integral) section of $H^0(\text{M}_{(\Delta)}^K \mathbf{O})_R, \Xi^*(\mathcal{E})_{\text{horz.}}^{\frac{a_0(0)}{2}})$ for any Δ .

Furthermore, the expansion (16) is invertible integrally, hence we may infer that $\Psi(F)^{-1}$ is a section¹⁰ of

$$H^0(\text{M}_{(\Delta)}^K \mathbf{O})_R, \Xi^*(\mathcal{E})_{\text{horz.}}^{-\frac{a_0(0)}{2}}).$$

It follows that $\Psi(F)$ cannot have any connected (=irreducible) component of the fibre above p in its divisor.

¹⁰Up to the problem of constants, this is $\Psi(-F)$.

To prove that $\Psi(F)$ is actually defined over $\mathbb{Z}_{(p)}$, i.e. lies in

$$H^0(\mathbf{M}(\Delta^K \mathbf{O}), \Xi^*(\mathcal{E})_{\text{horz.}}^{\frac{a_0(0)}{2}}),$$

we have to check the requested invariance property of the Fourier coefficients. We compute

$$a_{\iota, z', \lambda, \alpha, \rho, \sigma}({}^\tau \Psi(F)) = {}^\tau a_{\iota, z', \lambda, \alpha, \rho \iota(k_\tau)^{-1}, \sigma}(\Psi(F))$$

for all $\tau \in \text{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q})$.

By the transformation property (11), we are reduced to compute

$$a_{\iota, z', \lambda, \alpha, 1, \sigma}({}^\tau \Psi(F')) = {}^\tau a_{\iota, z', \lambda, \alpha, 1, \sigma}(\Psi(\omega(\iota(k_\tau))^{-1} F'))$$

for $F' = \omega(\rho)F$. We will, however, keep the letter F .

We have seen that, up to a constant of absolute value 1, the coefficients

$$a_{\iota, z', \lambda, \alpha, 1, \sigma}(\Psi(\omega(\iota(k_\tau))^{-1} F))$$

are the ones forming $\text{BP}(\omega(\iota(k_\tau))^{-1} F, I, z', \alpha, W)$. We have by definition

$$a_m(\omega(\iota(k_\tau))^{-1} F)(\rho) = a_m(F)(\iota(k_\tau)\rho).$$

However $\iota(k_\tau)$ acts trivial on $\lambda_{z'}$ because the latter is perpendicular to $I \oplus \langle z' \rangle$, and it acts by multiplication by k_τ on δ . Substituting $k_\tau^{-1}\delta$ for δ and using invariance of the measure ($k_\tau \in \hat{\mathbb{Z}}^*$!), we get the same expression, but all $\chi_\alpha(\langle z', \delta \rangle)$ changed to $\chi_\alpha(\langle z', k_\tau^{-1}\delta \rangle) = \chi_\alpha(k_\tau^{-1}\langle z', \delta \rangle) = \chi_\alpha(\langle z', \delta \rangle)^{\tau^{-1}}$. This shows:

$$a_{\iota, z', \lambda, \alpha, 1, \sigma}(\Psi(\omega(\iota(k_\tau))^{-1} F)) = \Lambda_\tau^{\tau^{-1}} a_{\iota, z', \lambda, \alpha, 1, \sigma}(\Psi(F))$$

with $|\Lambda_\tau| = 1$ and $\Lambda_\tau \in R^*$ (because $\Psi(F)$ is defined over R and invertible in the sense above).

Therefore

$${}^\tau f = {}^\tau \Lambda_\tau f$$

for all $\tau \in \text{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q})$, where Λ_τ locally constant invertible on $\mathbf{M}(\Delta^K \mathbf{O})_R$ with $|\Lambda_\tau|_\infty \equiv 1$ constant. From Hilbert 90 follows that there exists a Λ , locally constant invertible on $\mathbf{M}(\Delta^K \mathbf{O})_R$ such that

$${}^\tau (\Lambda f) = \Lambda f$$

(i.e. $({}^\tau \Lambda)/\Lambda = {}^\tau \Lambda_\tau$). That is, Λf is defined over $\mathbb{Z}_{(p)}$.

It remains to show 3. Without the Λ this is proven in [22] or [7] as an identity in \mathbb{R} , cf. beginning of this proof. The evaluation of

$$\int \log \|\Lambda f\| c_1(\Xi^*(\bar{\mathcal{E}}))^{m-2}$$

differs from the same expression for f instead of Λf by

$$\sum_{\alpha \in \pi_0(\mathbf{M}(\Delta^K \mathbf{O}))} \text{vol}(\alpha) \log(|\Lambda(\alpha)|_\infty).$$

Since $|\Lambda(\tau^{-1}\alpha)^\tau|_\infty = |\Lambda(\alpha)|_\infty$ we may write this as

$$\begin{aligned} & \frac{1}{\phi(M)} \sum_{\tau \in \text{Gal}(\mathbb{Q}(\zeta_M)|\mathbb{Q}), \alpha \in \pi_0(M({}^K\mathbf{O}))} \text{vol}(\alpha) \log(|\Lambda(\tau^{-1}\alpha)^\tau|_\infty) \\ &= \frac{1}{\phi(M)} \sum_{\tau \in \text{Gal}(\mathbb{Q}(\zeta_M)|\mathbb{Q}), \alpha \in \pi_0(M({}^K\mathbf{O}))} \text{vol}(\tau\alpha) \log(|\Lambda(\alpha)^\tau|_\infty) \\ &= \frac{1}{\phi(M)} \sum_{\alpha \in \pi_0(M({}^K\mathbf{O}))} \text{vol}(\alpha) \log(|N_{\mathbb{Q}(\zeta_M)|\mathbb{Q}}(\Lambda(\alpha))|_\infty) \equiv 0 \quad \text{in } \mathbb{R}^{(p)}. \end{aligned}$$

Here we used $\text{vol}(\tau\alpha) = \text{vol}(\alpha)$, which is true because $c_1(\Xi^*(\mathcal{E}))$ is Galois invariant. \square

We will later need some general lemmas about the line bundle of orthogonal modular forms.

Lemma 11.15. *With respect to the Hodge embedding (5.4)*

$$\mathbf{S}(L) \hookrightarrow \mathbf{H}(\mathbf{C}^+(L), \langle, \rangle_\delta),$$

the pullback of the canonical bundle $\Lambda^g \mathcal{L}$ via the ‘dual’ embedding

$$M^\vee(\mathbf{S}(L)) \rightarrow M^\vee(\mathbf{H}(\mathbf{C}^+(L), \langle, \rangle_\delta))$$

is, up to a bundle coming from a character $\text{GSpin}(L) \rightarrow \mathbb{G}_m$, and up to a tensor-power, the bundle \mathcal{E} (11.7).

Proof. Recall that the induced map on compact duals (9.2) is given by $\langle z \rangle \mapsto zw\mathbf{C}^+(L)$, where z is assumed to be primitive in $L_{\mathbb{Z}(p)}$ and w is *any* primitive vector with $\langle w, z \rangle = 1$. The fibre of \mathcal{L} at the image of $\langle z \rangle$ is given by the space $zw\mathbf{C}^+(L)$ and the fibre of \mathcal{E} is given by $\langle z \rangle$. By an explicit determination of the action of the stabilizer group in both cases the result follows. \square

Corollary 11.16. *The pullback of the bundle of Siegel modular forms $\Xi^*(\Lambda^g \mathcal{L})^{\otimes n}$ of some positive weight n is equal to some positive power of the bundle of orthogonal modular forms $\Xi^* \mathcal{E}^{\otimes n'}$. In particular $(\Xi^* \mathcal{E})$ is ample on every $M({}^K \mathbf{S}(L))$ and $M({}^K \mathbf{O}(L))$ and a suitable power has no base points on $M({}_\Delta^K \mathbf{O}(L))$.*

Proof. Follows from 11.15 and 10.5. The bundle of Siegel modular forms $\Xi^*(\Lambda^g \mathcal{L})$ is ample [12]. The ampleness of $\Xi^* \mathcal{E}$ follows from the fact that the Hodge embedding induces an embedding (up to a normlzization) of the corresponding Shimura varieties (6.5) for suitable levels and the fact that all $M({}^K \mathbf{S}(L))$ and $M({}^K \mathbf{O}(L))$ are related by finite etale maps that pullback the $\Xi^* \mathcal{E}$ ’s into each other. The assertion on base points is true, because a suitable power of $\Xi^*(\Lambda^g \mathcal{L})$ defines a morphism to the minimal compactification on every $M({}_\Delta^{K'} \mathbf{H}_g)$ [12, Theorem 2.3, (1)]. \square

We will compare the measure given by the volume form $c_1(\Xi^*\bar{\mathcal{E}})^{m-2}$, the highest power of the Chern form of the canonical Hermitian line bundle defined above with the quotient μ of the canonical volume form [17, 2.4] on $\mathrm{SO}(L_{\mathbb{R}})$ by the one giving $K_{\infty} = \text{stabilizer of some positive definite subspace } N \text{ in } L_{\mathbb{R}}$ the volume one.

Lemma 11.17.

$$2\lambda_{\infty}^{-1}(L;0)\mu = (c_1(\Xi^*\mathcal{E}, \Xi^*h))^{m-2}.$$

Proof. We will ignore henceforth all signs of volume forms, whenever they have no influence because we know that the 2 volume forms are positive and invariant. Choose an orthogonal basis of $L_{\mathbb{R}}$ with $\langle e_i, e_i \rangle = -1, i = 1, 2$ and $\langle e_i, e_i \rangle = 1, i > 2$. Set $z = \frac{1}{2}(e_3 - e_2), z' = \frac{1}{2}(e_3 + e_2)$. We have $\langle z, z \rangle = 0$ and $\langle z, z' \rangle = \frac{1}{2}$. Choose a base point $\langle \xi \rangle \in \mathbb{D}_{\mathbf{O}} \subset \mathbb{P}^1(L_{\mathbb{C}})$, $\xi = ie_1 + e_2$. It induces an isomorphism $\mathrm{SO}(L_{\mathbb{R}})/K_{\infty} \cong \mathbb{D}_{\mathbf{O}}$. The tangent space at ξ in $\mathbb{D}_{\mathbf{O}} \subset M^{\vee}(\mathbf{O}(L))(\mathbb{C})$ is canonically identified with

$$\mathrm{Hom}(\langle \xi \rangle_{\mathbb{C}}, \langle \xi \rangle_{\mathbb{C}}^{\perp} / \langle \xi \rangle).$$

The induced map

$$\mathrm{Lie}(\mathrm{SO}(L_{\mathbb{R}})) \rightarrow \mathrm{Hom}(\langle \xi \rangle_{\mathbb{C}}, \langle \xi \rangle_{\mathbb{C}}^{\perp} / \langle \xi \rangle_{\mathbb{C}})$$

is given by

$$\Lambda \mapsto \{\xi \mapsto \Lambda\xi \mod \langle \xi \rangle_{\mathbb{C}}\}.$$

On $\mathrm{Lie}(\mathrm{SO}(L_{\mathbb{R}}))$ we have the basis:

$$L_{ij} = \{e_i \wedge e_j\},$$

whose associated volume form is the canonical μ [17, Lemma 3.6]. The basis elements $L_{12}, L_{ij}, i, j \geq 3$ are mapped to 0. Its associated volume form is the canonical [17, 2.4] on K_{∞} [17, Lemma 3.6]. The induced volume form on $\mathbb{D}_{\mathbf{O}}$ hence is associated to

$$\bigwedge_{i=3}^m \tilde{L}_{1i}^* \wedge \bigwedge_{i=3}^m \tilde{L}_{2i}^* \in \bigwedge^{2(m-2)} \mathrm{Hom}(\langle \xi \rangle_{\mathbb{C}}, \langle \xi \rangle_{\mathbb{C}}^{\perp} / \langle \xi \rangle_{\mathbb{C}})^*,$$

where $\tilde{L}_{1i} = \{\xi \mapsto ie_i\}$ and $\tilde{L}_{2i} = \{\xi \mapsto e_i\}$ (wedge product over \mathbb{R} !). Now let $K = \langle e_1, e_4, \dots, e_m \rangle_{\mathbb{C}} = \langle z, z' \rangle_{\mathbb{C}}^{\perp}$. We calculate

$$\begin{aligned} 4\partial\bar{\partial}\log Y^2 &= \sum_{i,j \in I} \frac{\partial^2}{\partial y_i \partial y_j} \log(Y^2) \, dz_i \wedge d\bar{z}_j \\ &= \sum_{i,j \in I} -\frac{4\delta_i \delta_j y_i y_j}{Y^4} \, dz_i \wedge d\bar{z}_j + \sum_i \frac{2\delta_i}{Y^2} \, dz_i \wedge d\bar{z}_i \end{aligned}$$

where $K \ni Z = X + iY$, $I = \{1, 4, \dots, m\}$ and $\delta_1 = -1$ and $\delta_i = 1$ otherwise.

At the point ie_1 this yields:

$$(\mathrm{d} \mathrm{d}^c \log Y^2)^{m-2} = \left(\frac{1}{2\pi} \right)^{m-2} (m-2)! \mathrm{d} x_1 \wedge \mathrm{d} y_1 \wedge \cdots \wedge \mathrm{d} x_m \wedge \mathrm{d} y_m$$

(observe $\mathrm{d} \mathrm{d}^c = -\frac{1}{2\pi i} \partial \bar{\partial}$ and $\mathrm{d} z_i \wedge \mathrm{d} \bar{z}_i = 2i \mathrm{d} x_i \wedge \mathrm{d} y_i$).
Under the parametrization

$$\begin{aligned} K_{\mathbb{C}} &\rightarrow \mathrm{M}^{\vee}(\mathbf{O}(L))(\mathbb{C}) \\ Z &\mapsto Z + z' - \langle Z, Z \rangle z \end{aligned}$$

ie_1 is mapped to ξ . The tangent map T at this point is

$$Z \mapsto \{\xi \mapsto Z - 2i\langle Z, e_1 \rangle z \mod \langle \xi \rangle_{\mathbb{C}}\},$$

and we have: $T(e_1) = \tilde{L}_{23}, T(ie_1) = -\tilde{L}_{13}$ and $T(e_i) = \tilde{L}_{1i}, T(ie_i) = \tilde{L}_{2i}$ for $i \geq 4$.

This yields the volume form associated to

$$(m-2)! \left(\frac{1}{2\pi} \right)^{m-2} \bigwedge_{i=3}^m \tilde{L}_{1i}^* \wedge \bigwedge_{i=3}^m \tilde{L}_{2i}^* \in \bigwedge^{2(m-2)} \mathrm{Hom}(\langle \xi \rangle_{\mathbb{C}}, \langle \xi \rangle_{\mathbb{C}}^{\perp} / \langle \xi \rangle_{\mathbb{C}})^*.$$

Now observe that $K_{\infty} = \mathrm{SO}(N) \times \mathrm{SO}(N^{\perp})$ because K_{∞} is not allowed to change the orientation of N (the \mathbb{C} -oriented negative definite plane corresponding to $\langle \xi \rangle$). Hence the volume of K_{∞} is $2\pi \cdot \frac{1}{2} \prod_{j=1}^{m-2} 2^{\frac{\pi^{j/2}}{\Gamma(j/2)}}$ [17, Lemma 3.7, 2.].

This gives the factor $2 \prod_{j=2}^m \frac{1}{2} \frac{\Gamma(\frac{j}{2})}{\pi^{\frac{j}{2}}}$ which is $2\lambda_{\infty}^{-1}(L; 0)$. \square

12. The arithmetic of Borcherds input forms

12.1. In this section, we present and refine some results of [4]. We begin by recalling the notation of [loc. cit.]. Let $\Gamma \subset \mathrm{Sp}'_2(\mathbb{R})$ be a subgroup, commensurable with $\mathrm{Sp}'_2(\mathbb{Z})$ and with values in any representation V of $\mathrm{Sp}'_2(\mathbb{Z})$. Here Sp'_2 , as usual, is either $\mathrm{Sp}_2 = \mathrm{SL}_2$ or Mp_2 . Let k be a weight, half integral, if $\mathrm{Sp}' = \mathrm{Mp}$ and an integer, if $\mathrm{Sp}' = \mathrm{Sp}$.

Let

$$\mathrm{ModForm}(\Gamma, V, k)$$

be the space of modular forms of weight k for Γ of representation V , meromorphic at the cusps. Let

$$\mathrm{HolModForm}(\Gamma, V, k)$$

be the space of modular forms of weight k for Γ of representation ρ , which are holomorphic at the cusps.

12.2. Denote

$$\begin{aligned}\text{PowSer}(\Gamma) &= \bigoplus_{\kappa} \mathbb{C}[[q_{\kappa}]], \\ \text{Laur}(\Gamma) &= \bigoplus_{\kappa} \mathbb{C}[[q_{\kappa}]] [q_{\kappa}^{-1}], \\ \text{Sing}(\Gamma) &= \bigoplus_{\kappa} \mathbb{C}[[q_{\kappa}]] [q_{\kappa}^{-1}] / q_{\kappa} \mathbb{C}[[q_{\kappa}]],\end{aligned}$$

where κ runs through all cusps for Γ , and q_{κ} is a uniformizer at the cusp κ . $\text{Sp}_2'(\mathbb{Z})$ acts on these sets.

Let $\Gamma' \subset \Gamma$ be a subgroup of finite index, which acts trivial on V . We define

$$\text{PowSer}(\Gamma, V) = (\text{PowSer}(\Gamma') \otimes V)^{\Gamma},$$

where Γ acts on both factors.

Similarly we define $\text{Laur}(\Gamma, V)$ and $\text{Sing}(\Gamma, V)$. We have still (abstractly) a 1:1 correspondence between $f \in \text{PowSer}(\Gamma, V)$ (say) and a collection f_{κ} for each cusp of Γ (not Γ' !) where

$$f_{\kappa} = \sum_{m \in \mathbb{Q}_{\geq 0}} a_m q_{\kappa}^m, \quad (17)$$

and a_m lies in a sub-vector space V_m of V , depending on m .

For $\Gamma = \text{Sp}_2'(\mathbb{Z})$, we have explicitly

$$V_m = \{v \in V \mid \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v = e^{2\pi i m} v, \} \quad (18)$$

and in particular V_m is zero if $m \notin \frac{1}{N}\mathbb{Z}$ for some N , depending on Γ' .

There are maps

$$\begin{aligned}\lambda : \text{HolModForm}(\Gamma, V^*, k) &\rightarrow \text{PowSer}(\Gamma, V^*), \\ \lambda : \text{ModForm}(\Gamma, V, 2-k) &\rightarrow \text{Sing}(\Gamma, V),\end{aligned}$$

which in the representation (17) for the right hand side, this corresponds to taking Fourier series.

There is a non-degenerate pairing between $\text{PowSer}(\Gamma, V)$ and $\text{Sing}(\Gamma, V^*)$ given by

$$\langle f, \phi \rangle = \sum_{\kappa} \text{res}(\langle f_{\kappa}, \phi_{\kappa} \rangle q_{\kappa}^{-1} d q_{\kappa}).$$

This pairing is invariant under the operation of $\text{Mp}_2(\mathbb{Z})$, and hence gives a well defined pairing between the spaces in question.

For example if $\Gamma = \text{Mp}_2(\mathbb{Z})$ and standard parameter q , up to a scalar, this pairing is the same as

$$\langle f, \phi \rangle = \sum_m \langle a_{-m}, b_m \rangle,$$

for $f = \sum_m a_m q^m$ and $\phi = \sum_m b_m q^m$.

In [loc. cit., Theorem 3.1] it is sketched that one can deduce from Serre duality that

$$\lambda(\text{ModForm}(\text{Sp}'_2(\mathbb{Z}), V, 2-k)) = (\lambda(\text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), V^*, k)))^\perp, \quad (19)$$

and in [loc. cit., Lemma 4.3] this is refined to

$$\lambda(\text{ModForm}(\text{Sp}'_2(\mathbb{Z}), V, 2-k)_{\mathbb{Z}}) \otimes \mathbb{C} = (\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\lambda(\text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), V^*, k)))^\perp.$$

It is also shown that the last space has finite index in $\text{Sing}(\text{Sp}'_2(\mathbb{Z}), V)$.

Let L be a \mathbb{Z} -lattice of dimension m with non-degenerate quadratic form $Q_L \in \text{Sym}^2(L^*)$. We let $\text{Weil}(L^*/L) = \mathbb{C}[L^*/L]$ be the Weil representation of $\text{Sp}'_2(\mathbb{Z})$ and $2k = m$ modulo 2. If $V^* \subseteq \text{Weil}(L^*/L)^*$ is a sub-representation, which, as a linear subspace, is defined over \mathbb{Q} , Proposition 12.11 below states that

$$\lambda(\text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), V^*, k))$$

is actually Galois stable.

12.3. We start by a general discussion about the arithmetic of ‘vector valued modular forms’ for GL_2 here. We denote by \mathbb{Q}^{cycl} the direct limit of $\mathcal{O}_{\mathbb{G}_m, \mathbb{Q}[\frac{1}{N}]}$. It is isomorphic to \mathbb{Q}^{ab} but equipped with a compatible system of primitive N -th roots of unity $\{\zeta_N\}$. Define

$$\tau : \text{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$$

$$\tau : k \mapsto \tau_k := \text{rec} \circ \det(k)$$

which is characterized by $\tau_k \zeta_N = (\zeta_N)^{\det(k)}$.

Proposition 12.4. *For each $n \in \mathbb{Z}$ and an open subgroup $K \subseteq \text{GL}_2(\widehat{\mathbb{Z}})$, we have a natural exact functor from the category of τ -semi-linear representations of K (C.1) to vector bundles over $\text{M}(K\mathbf{H}_1)_{\mathbb{Q}}$ denoted*

$$V \mapsto \Xi^* \mathcal{L}_V^{\otimes n}.$$

For $K_1 \subseteq K_2 \subseteq \text{GL}_2(\widehat{\mathbb{Z}})$ open subgroups, we have a natural isomorphism:

$$H^0(\text{M}(K_1\mathbf{H}_1)_{\mathbb{Q}}, \Xi^* \mathcal{L}_V^{\otimes n}) \cong H^0(\text{M}(K_2\mathbf{H}_1)_{\mathbb{Q}}, \Xi^* \mathcal{L}_{\text{ind}_{K_1}^{K_2} V}^{\otimes n}).$$

Proof. The functor is defined as follows: Let N be such that the τ -semi-linear representation (V, ω) has a $\mathbb{Q}[\zeta_N]$ -structure V_N and $K(N)$ acts just via τ on $V \cong V_N \otimes_{\mathbb{Q}(\zeta_N)} \mathbb{Q}^{cycl}$.

We have the following obvious commutative diagram of extended Shimura data:

$$\begin{array}{ccc} K(N)\mathbf{H}_1 & \xrightarrow{(\text{id}, k)} & K(N)\mathbf{H}_1 \\ \downarrow (\det, 1) & & \downarrow (\det, 1) \\ K(N, \mathbb{G}_m)\mathbf{H}_0 & \xrightarrow{(\text{id}, \det(k))} & K(N, \mathbb{G}_m)\mathbf{H}_0 \end{array}$$

Applying the canonical model functor M to this diagram, we see, that $M(\det, 1)$ is compatible with the (Hecke) action of $\mathrm{GL}(\mathbb{Z}/N\mathbb{Z})$.

From the explicit description of the canonical model (cf. e.g. 7.13), we know that

$$M({}^{K(N, \mathbb{G}_m)}\mathbf{H}_0)_{\mathbb{Q}} \cong \mathrm{spec}(\mathbb{Q}(\zeta_N)),$$

and the Hecke action transports to the natural isomorphism

$$(\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}).$$

The τ -semi-linear representation V thus gives rise to a vector bundle \mathcal{V} with *right* $K/K(N)$ -action on

$$\mathrm{spec}(\mathbb{Q}(\zeta_N)),$$

which is compatible with the Hecke action (acting via $\tau = \mathrm{rec} \circ \det$). In detail: we let $k \in K$ act on V as $v \mapsto \omega(k)^{-1}v$ (right action). This action is compatible with the action on $\mathrm{spec}(\mathbb{Q}(\zeta_N))$ given by letting $k \in K$ act on a function $\alpha \in \mathbb{Q}(\zeta_N)$ as $\tau_k(\alpha)$, (i.e. the action $k \mapsto \mathrm{spec}(\tau_k)$ on $\mathrm{spec}(\mathbb{Q}(\zeta_N))$, if we have

$$\omega(k)(\alpha v) = \tau_k(\alpha)\omega(k)v.$$

This is precisely the condition of ω being τ -semi-linear.

We define $\Xi^* \mathcal{L}_V^{\otimes n}$ as a quotient of the bundle on $M({}^{K(N)}\mathbf{H}_1)_{\mathbb{Q}}$

$$\Xi^* \mathcal{L}^{\otimes n} \otimes M(\det, 1)^* \mathcal{V},$$

by the action of $K/K(N)$ on both factors. Since the action is compatible with the Hecke-action on $M({}^{K(N)}\mathbf{H}_1)_{\mathbb{Q}}$, it descends along the quotient $M({}^{K(N)}\mathbf{H}_1)_{\mathbb{Q}} \rightarrow M({}^K\mathbf{H}_1)_{\mathbb{Q}}$. Up to isomorphism, this bundle does not depend on the N and V_N chosen.

The induction formula is a variant of Shapiro's Lemma. \square

12.5. Fourier coefficients: We fix the boundary component \mathbf{B} (corresponding to $i\infty$ in the classical picture) with

$$P_{\mathbf{B}} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \quad \mathbb{D}_{\mathbf{B}} = \mathbf{H}_0 \times \mathbb{C}$$

with its standard splitting

$$P_{\mathbf{B}} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \rtimes \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$$

(corresponding to $0 \in \mathbb{C}$ in the classical picture). It gives rise to a boundary morphism

$$\mathbf{H}_0[0, \mathbb{Q}] \implies \mathbf{H}_1$$

with respect to which each $f \in H^0(M({}^{K_1}\mathbf{H}_1)_{\mathbb{C}}, \Xi^* \mathcal{L}_V^{\otimes n})$ has a Fourier expansion, depending on some fixed rational trivializing section $s \in H^0(M({}^K\mathbf{H}_0[0, \mathbb{Q}])_{\mathbb{Q}}, \Xi^* \mathcal{L})$ as in 8.2:

$$g(z, \rho) := \frac{f}{s}(z, \rho) = \sum_{k \in \frac{1}{N}\mathbb{Z}} a_{k, \alpha, \rho}(f) \exp(\alpha(kz)),$$

where $a_{k,\alpha,\rho} \in (V_{\mathbb{C}})_{k,\alpha}$, where

$$(V_{\mathbb{C}})_{k,\alpha} = \{v \in V_{\mathbb{C}} \mid \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v = \exp(\alpha(k))v\},$$

and α is determined by the connected component of \mathbf{H}^{\pm} that contains z . They satisfy (8.2)

$$a_{k,\alpha,\rho}(\tau f) = \tau a_{k,\alpha,\rho} \begin{pmatrix} k_{\tau}^{-1} & \\ & 1 \end{pmatrix} (f).$$

Now choose a \mathbb{Q} -structure $V_{\mathbb{Q}}$ on V such that ω gives rise to a cocycle $\omega' \in H^1(K, \mathrm{GL}_{\mathbb{Q}^{\mathrm{cycl}}}(V))$, where K acts via τ (see C.2). Any holomorphic function g

$$\mathbb{H}^{\pm} \times (\mathrm{GL}_2(\mathbb{A}^{(\infty)})) \rightarrow V_{\mathbb{C}}$$

which satisfies

$$g(\gamma z, \gamma \rho) = j(\gamma, z)^n g(z, \rho),$$

where j is the usual automorphy factor and

$$g(z, \rho k) = \omega'(k^{-1})^{\exp(\alpha(\det(\rho k)/N))} g(z, \rho) \quad (20)$$

for $\rho \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$, (where $\omega'(k^{-1})^{\exp(\alpha(\det(\rho k)/N))}$ means that $\omega'(k^{-1})$ acts on $V_{\mathbb{C}}$ by mapping ζ_N to $\exp(\alpha(\det(\rho k)/N))$), gives rise to a section

$$f = gs \in H^0(\mathrm{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{C}}, \Xi^* \mathcal{L}_V^{\otimes n}.$$

The condition (20) can be expressed purely in terms of Fourier coefficients:

$$a_{k,\alpha,\rho k}(g) = \omega'(k^{-1})^{\exp(\alpha(\det(\rho k)/N))} a_{k,\alpha,\rho}(g).$$

In particular, the Galois operation on f is determined by

$$a_{k,\alpha^+,1} \left(\frac{\tau f}{s} \right) = \omega' \left(\begin{pmatrix} k_{\tau} & \\ & 1 \end{pmatrix} \right)^{\exp(\alpha^+(\frac{1}{N}))} \cdot \tau(a_{k,\alpha^+,1} \left(\frac{f}{s} \right)),$$

where α^+ is one chosen isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}(1)$.

We have proven:

Proposition 12.6. *If $\det : K \rightarrow \widehat{\mathbb{Z}}^*$ is surjective and (V, ω) is a τ -semi-linear representation, the association $g \mapsto f := gs$ determines a bijection*

$$\mathrm{ModForm}(\Gamma, V, k) \rightarrow H^0(\mathrm{M}^{(K)} \mathbf{H}_1)_{\mathbb{C}}, (\Xi^* \mathcal{L})_V^{\otimes k},$$

where $\Gamma = K \cap \mathrm{SL}_2(\mathbb{Z})$. A section is determined by the Fourier coefficients

$$a_k(g) = a_{k,\alpha^+,1} \left(\frac{f}{s} \right).$$

Here α^+ corresponds to the connected component \mathbb{H}^+ chosen as domain for the classical modular form g . f is rational, if we have:

$$a_k(g) = \omega' \left(\begin{pmatrix} k_{\tau} & \\ & 1 \end{pmatrix} \right)^{\exp(\alpha^+(\frac{1}{N}))} \cdot \tau(a_k(g)).$$

It is crucial that, if (V, ω) is given by a $\mathbb{Q}(\zeta_N)$ -representation of $\mathrm{SL}(\mathbb{Z}/N\mathbb{Z})$, say, for the theorem to make sense, the representation must extend τ -semi-linearly to $\mathrm{GL}(\mathbb{Z}/N\mathbb{Z})$.

Example 12.7. Consider the special case $K = K_0(p)$, p an odd prime. Let ω be the τ -semi-linear representation $\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto \chi(d) \cdot \mathrm{rec}(ad)$ defined on $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ for the unique non-trivial character $\chi : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \{\pm 1\}$ of order 2. The action of Galois on the Fourier coefficients is the natural, untwisted one. We have a decomposition (C.5)

$$\mathrm{ind}_{K_0(p)}^{K(1)}(\omega) = V^+ \oplus V^-$$

as τ -semi-linear representations, and hence Propositions 12.4 and 12.6 yield

$$\mathrm{ModForm}(\Gamma_0(k), \chi, k) = H^0(\mathrm{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{C}}, (\Xi^* \mathcal{L})_{V^+}^{\otimes k} \oplus H^0(\mathrm{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{C}}, (\Xi^* \mathcal{L})_{V^-}^{\otimes k}.$$

This decomposition has been described by Hecke. We get here a priori that it is actually defined over \mathbb{Q} . See also 12.10 below.

Let L be a lattice of even dimension with non-degenerate quadratic form.

Lemma 12.8. The Weil representation $\mathrm{Weil}(L^*/L)$, identifying $\exp(2\pi i/N)$, as chosen in the underlying characters, with the distinguished $\zeta_N \in \mathbb{Q}^{cycl}$, gives rise to a τ -semi-linear representation, where the matrix $\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ acts as $\mathrm{rec}(\alpha)$. The dual of the Weil representation gives rise to a τ -semi-linear representation, too.

Note that this τ -semi-linear representation does not involve any choices anymore and does depend only on L .

Proof. That the above defines indeed a τ -semi-linear representation boils down to the equation:

$$\omega\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} k \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{rec}(\alpha)(\omega(k)) \quad (21)$$

for the Weil representation (cf. 11.1). This is checked as follows: The operators $\varphi \mapsto (x^* \mapsto \varphi({}^t \alpha x^*))$ are Galois invariant. If we act on the operator $\varphi \mapsto (x^* \mapsto \exp(2\pi i Q_L(x^*) \beta) \varphi(x^*))$ by τ_α , we get the same as by substituting β by $\alpha\beta$. If we act on the operator $\varphi \mapsto (x^* \mapsto \int_{L_{\mathbb{A}(\infty)}} \varphi(\gamma x) \exp(-2\pi i x^* x) dx)$ by τ_α , we get the same as by substituting γ by $\alpha^{-1}\gamma$, provided dx is chosen Galois invariant (e.g. such that $L_{\widehat{\mathbb{Z}}}$ has a volume in \mathbb{Q}). Now an actual operator $\omega(r''(g_l(\alpha)))$ differs by those considered by a $\frac{\Upsilon_f(\alpha Q_L)}{\Upsilon_f(Q_L)}$ in the first case, which is a sign because m is even, hence Galois invariant. In the third case $\omega(r''(d(\gamma)))$ differs by $c(\gamma) := \frac{\Upsilon_f(\gamma Q_L)}{|\gamma|^{\frac{1}{2}}}$, where $|\gamma|$ is calculated with respect to the chosen measure and its dual. Now consider the equation $(u(-\gamma^{-1})d(\gamma))^3 = 1$. Since $c(\gamma)^2 \in \mathbb{Q}$, it shows $c(\alpha^{-1}\gamma) = \tau_\alpha c(\gamma)$. \square

12.9. Consider the Weil representation $\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$ as a τ -semi-linear representation of $\text{SL}_2(\widehat{\mathbb{Z}})$. $K_0(N)$ acts on $\chi_{L_{\widehat{\mathbb{Z}}}}$ by

$$\begin{pmatrix} a & * \\ & d \end{pmatrix} \chi_{L_{\widehat{\mathbb{Z}}}} = \Upsilon(dQ_L) \Upsilon(Q_L)^{-1} \chi_{L_{\widehat{\mathbb{Z}}}}$$

which is a character, valued in μ_2 . Call this character ε .

We have hence maps

$$\alpha : \varepsilon \rightarrow \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}),$$

and we have also the evaluation at 0 map:

$$\beta : \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \rightarrow \varepsilon.$$

Using the adjunctions, we get maps of τ -semi-linear representations

$$\tilde{\alpha} : \text{ind}_{K_0(N)}^{K(1)} \varepsilon \rightarrow \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}),$$

and

$$\tilde{\beta} : \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \rightarrow \text{ind}_{K_0(N)}^{K(1)} \varepsilon.$$

We have $\tilde{\beta} \circ \tilde{\alpha} = \text{id}$.

Using the functor, we get maps

$$\alpha' : H^0(\text{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{Q}}, \Xi^* \mathcal{L}_{\text{ind}_{K_0(N)}^{K(1)} \varepsilon}^{\otimes k} \rightarrow H^0(\text{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{Q}}, \Xi^* \mathcal{L}_{\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})}^{\otimes k}$$

$$\beta' : H^0(\text{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{Q}}, \Xi^* \mathcal{L}_{\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})}^{\otimes k} \rightarrow H^0(\text{M}^{(K(1))} \mathbf{H}_1)_{\mathbb{Q}}, \Xi^* \mathcal{L}_{\text{ind}_{K_0(N)}^{K(1)} \varepsilon}^{\otimes k}.$$

We get (this is true more generally) that the operator $\tilde{\beta} \circ \tilde{\alpha}$ is given by convolution with the function

$$\varphi(g) := \beta \circ \omega(g) \circ \alpha,$$

that is

$$\tilde{\beta}(\tilde{\alpha}(f))(h) = \sum_{a \in K_0(p) \backslash K(1)} \beta(\omega(ha^{-1})\alpha(f(a))).$$

Example 12.10. The following was inspired by [2].

Resuming example 12.7 above, it follows directly from the formulæ of the Weil-representation (11.2), that this function φ corresponding to $\tilde{\beta} \circ \tilde{\alpha}$ is determined by

$$\varphi(1) = 1 \quad \varphi(S) = \frac{\sqrt{-1}^{2-\frac{m}{2}}}{\sqrt{p}}.$$

Using the dichotomy between $p \bmod 4$, the type of $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$, and $\frac{m}{2} \bmod 4$, see [2, §2], we get

$$\varphi(S) = \begin{cases} \frac{1}{\sqrt{p^*}} & \text{if } L_{\mathbb{Z}}^*/L_{\mathbb{Z}} \text{ represents the squares mod } p, \\ -\frac{1}{\sqrt{p^*}} & \text{if } L_{\mathbb{Z}}^*/L_{\mathbb{Z}} \text{ represents the non-squares mod } p, \end{cases}$$

where $p^* = p$ if $p \equiv 1 \pmod{4}$ and $p^* = -p$, otherwise. Here $\sqrt{-1}$ is the distinguished primitive 4-th root of unity in \mathbb{Q}^{cycl} .

Therefore $\tilde{\beta} \circ \tilde{\alpha}$ is a projector (C.5) onto V^+ , resp. V^- , and thus $\tilde{\alpha}$ yields an isomorphism

$$V^\pm \cong \text{Weil}(L^*/L)^+$$

according to the isomorphism class of L^*/L . We therefore see from condition (18) and the explicit formulæ for the Weil-representation that for $\lambda \notin \mathbb{Z}$, we have

$$\begin{aligned} (V_{\mathbb{C}}^+)_{\lambda} &\cong \begin{cases} \mathbb{C} & pk \text{ is a square mod } p, \\ 0 & p\lambda \text{ is a non-square mod } p, \end{cases} \\ (V_{\mathbb{C}}^-)_{\lambda} &\cong \begin{cases} \mathbb{C} & pk \text{ is a non-square mod } p, \\ 0 & p\lambda \text{ is a square mod } p. \end{cases} \end{aligned}$$

Recall that the Fricke involution W_q is given by

$$W_q(f)(\tau) := q^{\frac{k}{2}}(q\tau)^{-k} f\left(\begin{pmatrix} & -1 \\ q & \end{pmatrix} \tau\right).$$

In terms of the corresponding $\tilde{f} \in \text{ModForm}(\text{SL}_2(\mathbb{Z}), \text{ind}_{\Gamma_0(0)}^{\text{SL}_2(\mathbb{Z})} \chi, k)$, we get

$$W_q(f)(\tau) = (q^{\frac{k}{2}} \rho\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\right) \tilde{f}(q\tau))(1).$$

Hence $\text{ModForm}(\text{SL}_2(\mathbb{Z}), V_{\mathbb{C}}^{\pm}, k)$ is the space of those $f \in \text{ModForm}(\Gamma_0(q), \chi, k)$ such that the Fourier coefficients a_{λ} of $W_q(f)$, for $(\lambda, q) = 1$ are 0 if λ is a non-square (resp. square) mod p . This is (for $W_q(f)$) the classical condition.

Proposition 12.11. *Let k be a (half-)integral weight and Sp'_2 either Sp_2 or Mp_2 according to whether $k \in \mathbb{Z}$ or $k \in \frac{1}{2} + \mathbb{Z}$ and suppose $2k \equiv m$ modulo 2. For $f \in \text{ModForm}(\text{Sp}'_2, \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, k)$ with Fourier expansion*

$$f = \sum_{k \in \mathbb{Q}} a_k q^k,$$

$a_k \in \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*_k$, we have for $\tau \in \text{Aut}(\mathbb{C})$ and

$${}^{\tau}f := \sum_{k \in \mathbb{Q}} {}^{\tau}a_k q^k,$$

that ${}^{\tau}f \in \text{ModForm}(\text{Sp}'_2, \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, k)$, too.

Proof. We first reduce to the case $k \in \mathbb{Z}$. For half integral weight, i.e. m odd, consider the space $L'_{\mathbb{Z}} = L_{\mathbb{Z}} \oplus \langle 1 \rangle$. We have

$$\text{Weil}((L'_{\mathbb{Z}})^*/(L'_{\mathbb{Z}}))^* = \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^* \otimes \text{Weil}(\langle 1 \rangle^* / \langle 1 \rangle^*).$$

We have the classical theta function

$$\theta \in \text{HolModForm}(\text{Mp}_2, \text{Weil}(< 1 >^* / < 1 >^*, \frac{1}{2}))$$

(with integral Fourier coefficients!). Multiplication with it yields a map

$$\text{ModForm}(\text{Mp}_2, \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, k) \rightarrow \text{ModForm}(\text{Mp}_2, \text{Weil}((L'_{\mathbb{Z}})^*/L'_{\mathbb{Z}})^*, k + \frac{1}{2}),$$

which commutes with the Galois action on Fourier coefficients (because θ has integral Fourier coefficients). Now, if $f \in \text{ModForm}(\text{Sp}'_2, \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, k)$ is given, we get (assuming the integral case) that

$${}^{\tau}f \otimes \theta = ({}^{\tau}f) \otimes \theta \in \text{ModForm}(\text{Sp}'_2, \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, k + \frac{1}{2}),$$

or, in other words:

$$({}^{\tau}f)(gz) \otimes \theta(g\tau) = j(g, \tau)^{2k+1} (\omega_L(g)({}^{\tau}f)(g\tau)) \otimes (\omega_{<1>}(g)\theta(g\tau)),$$

where $j(g, \tau)$ is the fundamental cocycle of weight $\frac{1}{2}$ on $\text{Mp}_2(\mathbb{Z}) \times \mathbb{H}$.

For a dense set of τ , $\theta(\tau)$ and $\theta(g\tau)$ are not zero; hence ${}^{\tau}f$ must converge, too, and

$$({}^{\tau}f)(gz) = j(g, \tau)^{2k} (\omega_L(g)({}^{\tau}f)(g\tau)),$$

using that θ is in $\text{HolModForm}(\text{Mp}_2, \text{Weil}(< 1 >^* / < 1 >^*, \frac{1}{2}))$. Hence ${}^{\tau}f$ lies in $\text{ModForm}(\text{Mp}_2, \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, k)$.

Now we may assume that $k \in \mathbb{Z}$ and $\text{Sp}'_2(\mathbb{Z}) = \text{Sp}_2(\mathbb{Z})$. Then the statement of the theorem follows from Lemma 12.9 and Propositions 12.4 and 12.6. \square

12.12. We resume the notation from section 12.2. Assume now that $n \geq 3$ — or — $n = 2$ and Witt rank of $L_{\mathbb{Q}} = 1$. With the results above, we are able to construct Borcherds products $\Psi(F)$ whose divisor

$$\text{div}(\Psi(F)^2) = \sum_{m \in \mathbb{Q}_{<0}} Z(L_{\mathbb{Z}}, < -m >, a_m; K) + D_{\infty}$$

satisfies certain special properties (see Theorems 12.14–12.17 below).

Lemma 12.13. *Let $L_{\mathbb{Z}}$ be a lattice of signature $(m-2, 2)$, $m \geq 4$. Let Weil_0 be a sub-representation of the Weil representation $\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$, which is, as a subspace, defined over \mathbb{Q} . Let $\{M(m)\}_{m \in \mathbb{Q}_{>0}}$ be a collection of sub-vector spaces $M(m) \subseteq \text{Weil}_{0,m}^*$ (for the notation, see 12.9), such that $\sum_m \dim(M(m)^{\perp}) \rightarrow \infty$.*

Assume that any modular form in $\text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^, \frac{m}{2})$ whose Fourier coefficients satisfy $a_m \in M(m)$ for all $m > 0$, vanishes.*

1. *There is an $F \in \text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0, 2 - \frac{m}{2})$ with Fourier expansion:*

$$\lambda(F) = \sum_{m \in \mathbb{Q}} a_m q^m$$

with $a_0(0) \neq 0$ and all $a_m \perp M(-m)$, $m < 0$,

2. Let $0 < l \in \mathbb{Q}$ and $a_{-l} \in \text{Weil}_{0,-l}$ with $a_{-l} \notin M(l)$ be given.
There is an $F \in \text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0, 2 - \frac{m}{2})$ with Fourier expansion:

$$\lambda(F) = \sum_{m \in \mathbb{Q}} a_m q^m$$

with $a_0(0) \neq 0$, all $a_{-m} \perp M(m)$, $m > 0$, $m \neq l$ and a_{-l} is the given one.

In both cases we may assume $a_m \in \mathbb{Z}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$ for all $m \in \mathbb{Q}$.

Proof. (Compare [8, Lemma 4.11])

1. Let $\text{Sing}_M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0) \subset \text{Sing}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)$ be the subspace, defined by the conditions $a_{-m} \perp M(m)$, $m > 0$. Obviously $\text{Sing}_M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)^\perp$ is the subspace $\text{PowSer}^M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*)$, defined by the conditions $a_m^* \in M(m)$ for all $m > 0$ and $c_0^* = 0$.

Since $\lambda(\text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0, 2 - \frac{m}{2})_{\mathbb{Q}}) \otimes \mathbb{C}$ has f. ind. in $\text{Sing}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)$ and $\sum_m \dim(M(m)^\perp) \rightarrow \infty$, we have

$$\lambda(\text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0, 2 - \frac{m}{2})_{\mathbb{Q}}) \cap \text{Sing}_M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)_{\mathbb{Q}} \neq 0.$$

We want to show that the application $[a_0(0)] : \text{Sing}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)_{\mathbb{Q}} \rightarrow \mathbb{Q}$ does not vanish on this intersection.

Now the duality between $\text{PowSer}(\dots)$ and $\text{Sing}(\dots)$ is non-degenerate, therefore:

$$\begin{aligned} & \left(\lambda(\text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0, 2 - \frac{m}{2})_{\mathbb{Q}}) \cap \text{Sing}_M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)_{\mathbb{Q}} \right)^\perp \\ &= \lambda(\text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*, \frac{m}{2}))^{\text{Aut}(\mathbb{C})} + \text{PowSer}^M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*)_{\mathbb{Q}}. \end{aligned}$$

Here we used Proposition 12.11, stating that $\lambda(\text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*, \frac{m}{2}))$ is a Galois invariant subspace because by assumption Weil_0^* is, as a subspace, defined over \mathbb{Q} .

Let us assume that there is a relation

$$[a_0(0)] = \lambda(f) + \phi,$$

where $f \in \text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*, \frac{m}{2})$ and ϕ is in $\text{PowSer}^M(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*)$. From this follows that f is a modular form, whose coefficients a_m^* , $m > 0$, satisfy $a_m^* \in M(m)$. Therefore $f = 0$ by assumption, a contradiction.

2. Since $a_{-l} \notin M(l)$, we find a $a_l^* \in M(l)$ such that $a_l^* a_{-l} \neq 0$. We have to show that the element $[a_l^* q^l] : \text{Sing}(\text{Sp}'_2(\mathbb{Z}), \rho)_{\mathbb{Q}} \rightarrow \mathbb{Q}$ does not vanish on the intersection

$$\lambda(\text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0, 2 - \frac{m}{2})_{\mathbb{Q}}) \cap \text{Sing}_{\widetilde{M}}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0)_{\mathbb{Q}},$$

where $\widetilde{M} = M$ except that $\widetilde{M}(l)$ has been set to 0.

For if this is not the case, we can add an appropriate element not being in its kernel to our original F to get the result. Suppose that $[a_l^* q^l]$ vanishes. Then we get a relation:

$$[a_l^* q^l] = \lambda(f) + \phi,$$

where $f \in \text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*, \frac{m}{2})$ and $\phi \in \text{PowSer}^{\widetilde{M}}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}_0^*)$. Therefore $f = 0$ as above, a contradiction.

The statement about integrality follows from the fact that all components of the constructed f s are in fact modular forms for $\Gamma(N)$ for some N . For integral weight it is well-known that their Fourier coefficients have a bounded denominator. Multiplication with the common denominator yields a form with integral coefficients. For half-integral weight, use the reduction to integral weight method of the proof of Proposition 12.11. \square

We will now show several theorems, stating the existence of Borchers lifts, whose divisor has special properties. This is an essential ingredient in the calculation of arithmetic volumes later. To not interrupt the discussion, we refer to the appendix A for several elementary lemmas on quadratic forms which are needed and to appendix B for several facts about vanishing of modular forms with sparse Fourier coefficients.

Theorem 12.14. 1. Let $L_{\mathbb{Z}(p)}$ be an isotropic lattice **of signature** $(m - 2, 2)$, $m \geq 4$. There is a saturated lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}(p)}$ and a $F \in \text{ModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), 2 - \frac{m}{2})$ with integral Fourier coefficients, such that $\Psi(F)$ has non-zero weight, all occurring $Z(L_{\mathbb{Z}}, < -m >, a_m; K)$, $m < 0$ in $\text{div}(\Psi(F))$ are p -integral, i.e. consist of canonical models of Shimura varieties $M(K\mathbf{O}(L'_{\mathbb{Z}(p)}))$, for various lattices $L'_{\mathbb{Z}(p)}$ with unimodular (at p) quadratic form.

2. We find an F above, such that $\text{div}(\Psi(F))$ contains, in addition, precisely one $Z(L_{\mathbb{Z}}, < m >, \kappa; K)$, $p|m$ with non-zero multiplicity.

Proof. 1. Choose any saturated lattice of the form $L_{\mathbb{Z}} = H \oplus L'_{\mathbb{Z}}$ in $L_{\mathbb{Z}(p)}$. Let

$$M(m) = \begin{cases} 0 & |m|_p = 1, \\ \text{Weil}_m^* & |m|_p \neq 1. \end{cases}$$

Then all $Z(L_{\mathbb{Z}}, < -m >, a_m; K)$ with $a_m \perp M(m)$ consist of p -integral canonical models of Shimura varieties $M(K\mathbf{O}(L'_{\mathbb{Z}(p)}))$ because v^\perp , for any $v \in L_{\mathbb{Z}_p}$ with $Q_L(v) = m$, $|m|_p = 1$, is unimodular.

In view of Theorem 11.14, to construct the required Borchers form, by Lemma 12.13, 1., we have to show that any modular form

$$f \in \text{HolModForm}(\text{Sp}'_2(\mathbb{Z}), \text{Weil}^*, \frac{m}{2}),$$

whose Fourier coefficients are supported only on M , vanishes. But every component of f is (in particular) a modular form for some $\Gamma(N)$, $p \nmid N$. It vanishes by Lemma B.1.

For 2., use Lemma 12.13, 2. \square

Theorem 12.15. 1. Let $L_{\mathbb{Z}_{(p)}}$ be a lattice **of signature** $(3, 2)$ and **Witt rank 1**. Up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$, there is a saturated lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ and a $F \in \text{ModForm}(\text{Mp}_2(\mathbb{Z}), \text{Weil}, 2 - \frac{m}{2})$ with integral Fourier coefficients, such that $\Psi(F)$ has non-zero weight, all occurring $Z(L_{\mathbb{Z}}, \langle -m \rangle, a_m; K)$ in $\text{div}(\Psi(F))$ are p -integral, i.e. consist of canonical models of Shimura varieties $M(K\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$, for various lattices $L'_{\mathbb{Z}_{(p)}}$ with unimodular (at p) quadratic form (i.e. with K admissible), such that $L'_{\mathbb{Q}}$ has signature $(2, 2)$ and Witt rank 1.

2. Up to multiplication of $Q_{L'}$ with a scalar $\in \mathbb{Z}_{(p)}^*$, for every $L'_{\mathbb{Z}_{(p)}}$ **of signature** $(2, 2)$, **Witt rank 0**, we find a lattice $L_{\mathbb{Z}_{(p)}}$ of signature $(3, 2)$, Witt rank 1, and an F as in 1., such that in $\text{div}(\Psi(F))$ occurs, in addition to the subvarieties above, precisely one $Z(L, \langle l \rangle, \kappa; K)$ with non-zero coefficient, consisting of canonical models of Shimura varieties $M(K_i\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$ (with the given $L'_{\mathbb{Z}_{(p)}}$) for various different admissible K_i 's.

Proof. 1. By Lemma A.3, we may multiply Q_L by a scalar in such a way that there is a saturated lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ with cyclic $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ of order $2D'$, where D' is square-free, of the form

$$L_{\mathbb{Z}} = H \oplus L'_{\mathbb{Z}}.$$

By 11.2, we know that $\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$ decomposes

$$\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) = \bigotimes_{l|D} \text{Weil}(L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l}),$$

and for each l we have a decomposition

$$\text{Weil}(L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l}) = \text{Weil}(L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l})^+ \oplus \text{Weil}(L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l})^-,$$

into irreducible representations, here $L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l}$ cyclic of odd prime order (respectively of order 2 or 4) is used. (If the order is 2, $\text{Weil}(L_{\mathbb{Z}_2}^*/L_{\mathbb{Z}_2})^-$ is zero).

We will work with the *irreducible* representation

$$\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+ := \bigotimes_l \text{Weil}(L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l})^+,$$

with product basis build from the bases $(\chi_{\kappa} + \chi_{-\kappa}) \in \text{Weil}(L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l})^+$ for $\kappa \in (L_{\mathbb{Z}_l}^*/L_{\mathbb{Z}_l})/(\pm 1)$. There will be a basis vector of the form

$$\gamma := \sum \chi_{\kappa},$$

where the sum runs only over *primitive* $\kappa \in L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$.

Let

$$M(m) = \begin{cases} (\mathbb{C}\gamma)^{\perp} & |m|_p = 1, \\ (\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+)_m^* & |m|_p \neq 1. \end{cases}$$

Now for $v \in L_{\mathbb{Q}} \cap \kappa$, with κ such that χ_{κ} occurs in γ (in particular κ primitive), and $|Q_L(v)|_p = 1$, v^{\perp} is unimodular. It is also isotropic because by Lemma A.3 m is already represented by $L'_{\widehat{\mathbb{Z}}}$.

In view of Theorem 11.14 again, to construct the required Borchers form, by Lemma 12.13, 2., we have to show that any modular form

$$f \in \text{HolModForm}(\text{Mp}_2(\mathbb{Z}), (\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+)^*, \frac{m}{2})$$

with Fourier coefficients supported only on M vanishes. Note that $\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+$ is defined over \mathbb{Q} as a subvectorspace.

Now $\gamma \circ f$ vanishes by B.1. Since $\text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+$ is irreducible, Lemma B.3 tells us $f = 0$.

2. Use A.5 to construct the lattice $L_{\mathbb{Z}}$, and $Z(L_{\mathbb{Z}}, \langle x \rangle, \chi_{L_{\mathbb{Z}}}; K)$ consists obviously of the required models. Since $\chi_{L_{\mathbb{Z}}} \not\subset \gamma^{\perp}$, apply Lemma 12.13, 2. \square

Theorem 12.16. 1. Let $L_{\mathbb{Z}(p)}$ be an **isotropic lattice of signature** $(2, 2)$ **and discriminant** q , q a prime $\equiv 1 \pmod{4}$.

Up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$, there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}(p)}$ and a $F \in \text{ModForm}(\text{Sp}_2(\mathbb{Z}), \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), 2 - \frac{m}{2})$ with integral Fourier coefficients, such that $\Psi(F)$ has non-zero weight, all occurring $Z(L, \langle -m \rangle, a_m; K)$ in $\text{div}(\Psi(F))$ are p -integral, i.e. consist of canonical models of Shimura varieties $M^{(K)}(\mathbf{O}(L'_{\mathbb{Z}(p)}))$, for various lattices $L'_{\mathbb{Z}(p)}$ with unimodular quadratic form, such that L' has signature $(1, 2)$ and Witt rank 1.

2. For every $L'_{\mathbb{Z}(p)}$ of **signature** $(1, 2)$, **Witt rank 0**, up to multiplication of $Q_{L'}$ with a scalar $\in \mathbb{Z}_{(p)}^*$, we find a lattice $L_{\mathbb{Z}}$ of signature $(2, 2)$, Witt rank 1 and discriminant q , q a prime $\equiv 1 \pmod{4}$ (as in i) and an F as in 1. such that in $\text{div}(\Psi(F))$ occurs, in addition to the subvarieties above, precisely one $Z(L, \langle l \rangle, \varphi; K)$ with non-zero coefficient, consisting of canonical models of Shimura varieties $M^{(K_i)}(\mathbf{O}(L'_{\mathbb{Z}(p)}))$ (with the given $L_{\mathbb{Z}(p)}$) for various different admissible K_i 's.

Proof. 1. Let Weil^+ be as in the last theorem.

By Lemma A.3, we may multiply Q_L by a scalar such that there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}(p)}$, with cyclic $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ of order q , of the form

$$L_{\mathbb{Z}} = H \oplus \langle x^2 + xy + \frac{1-q}{4}y^2 \rangle.$$

A $Z(L, \langle m \rangle, \kappa; K)$ (for admissible K) consists of Shimura varieties of the required form, if v^{\perp} for $v \in \pm\kappa + L_{\widehat{\mathbb{Z}}}$, $Q_L(v) = m$ is isotropic. This is the case, if and only if m is represented by $\langle x^2 + xy + \frac{1-q}{4}y^2 \rangle$. Hence define

$$M(m) := \begin{cases} (\text{Weil}^+)_m^* & |m|_p \neq 1, \\ (\text{Weil}^+)_m^* \cap R(m)^{\perp} & |m|_p = 1, \end{cases}$$

where $R(m) = \{f \in \text{Weil}_{0,m} \mid \exists v \in \langle x^2 + xy + \frac{1-q}{4}y^2 \rangle_{\mathbb{A}(\infty)} : f(v) \neq 0, Q_L(v) = m\}$.

We know (12.10) that $(\text{Weil}^+)^*_{\frac{1}{q}}$ is zero, if $\chi_q(j) = -1$. The definition of $M(m)$

and A.4 imply that for *primes* $p' \neq p$ with $\chi_q(p') = 1$, $M(\frac{p'}{q})$ is also zero.

In view of Theorem 11.14 again, to construct the required meromorphic modular form, by Lemma 12.13, we have to show that any modular form $f \in \text{HolModForm}(\text{Sp}_2(\mathbb{Z}), (\text{Weil}^+)^*, \frac{m}{2})$ with Fourier coefficients supported only on M vanishes. From the remarks in 12.10 it follows that f' defined by $f'(\tau) := \rho\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)f(q\tau)(1)$ is a scalar multiple of $W_q(f(1))$, hence a modular form for

$\Gamma_0(q)$ again. For primes $p' \neq p$, $M(\frac{p'}{q})$ is zero, hence the Fourier coefficient $a_{p'}$ of f' is zero. Similarly for $(n, q) = 1$, $\chi_p(n) = -1$, a_n is zero. Therefore the vanishing of f' now follows from Proposition B.2 and that of f by Lemma B.3. 2. Use A.2 to construct the lattice $L_{\mathbb{Z}}$ and $Z(L_{\mathbb{Z}}, \langle x \rangle, \chi_{L_{\mathbb{Z}}}; K)$ consists obviously of the required models. Since $\chi_{L_{\mathbb{Z}}} \not\perp R(m)^{\perp}$, apply Lemma 12.13, 2. \square

Theorem 12.17. *Let $L_{\mathbb{Z}(p)}$ be a **lattice of signature** $(3, 2)$ **of Witt rank 2**. There is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}(p)}$, such that up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$, there is an $F \in \text{ModForm}(\text{Mp}_2(\mathbb{Z}), \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), 2 - \frac{m}{2})$ with integral Fourier coefficients, such that $\Psi(F)$ has non-zero weight, $\text{div}(\Psi(F))$ consist of exactly one $Z(L_{\mathbb{Z}}, \langle -l \rangle, a_l; K)$ with non-zero coefficient, which itself consists of canonical models of a Shimura varieties $M(K\mathbf{O}(L'_{\mathbb{Z}(p)}))$, for any (a priori) given lattice $L'_{\mathbb{Z}(p)}$ of signature $(2, 2)$, Witt rank ≥ 1 .*

Proof. Define $L_{\mathbb{Z}(p)} = L'_{\mathbb{Z}(p)} \perp \langle x \rangle$, where $-x$ is represented by $L'_{\mathbb{Z}(p)}$ and $|x|_p = 1$. Up to multiplication of Q_L by a scalar, we find a saturated lattice $L_{\mathbb{Z}} = H^2 \perp \langle 1 \rangle \subset L_{\mathbb{Z}(p)}$. $\text{Weil} = \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$ is irreducible in this case. W.l.o.g x is assumed to be represented by $L_{\widehat{\mathbb{Z}}}$. Define

$$M(m) = \begin{cases} \text{Weil}_m^* & m \neq x, \\ 0 & m = x. \end{cases}$$

In view of Theorem 11.14 again, to construct the required meromorphic modular form, by Lemma 12.13, we have to show that any modular form $f \in \text{HolModForm}(\text{Mp}_2(\mathbb{Z}), \text{Weil}^*, \frac{5}{2})$ with Fourier coefficients supported only on M vanishes. This follows because it is well-known that

$$\text{HolModForm}(\text{Mp}_2(\mathbb{Z}), \text{Weil}^*, \frac{5}{2})$$

is generated by the Eisenstein series whose coefficient a_x is non-zero because x is represented by $L_{\widehat{\mathbb{Z}}}$. \square

13. Borchers' products and Arakelov geometry

13.1. Let $L_{\mathbb{Q}}$ be a f.d. vector space with non-degenerate quadratic form Q_L of signature $(m - 2, 2)$ with $m \geq 4$, and assume the Witt rank to be 1 if $m = 4$. Let $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ be a lattice of the form $L_{\mathbb{Z}} = H \perp L'_{\mathbb{Z}}$.

Take a modular form F as in 11.3 with Fourier expansion

$$F(\tau) = \sum_{m \in \mathbb{Q}} a_m q^m,$$

where $a_m \in \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \subset S(L_{\mathbb{A}(\infty)})$, with $a_0(0) \neq 0$.

In this section we prove a relation (Theorem 13.2, 2.) between arithmetic volumes of different Shimura varieties of orthogonal type which involves Borcherds lifts — in particular their integral (11.14, 3). It will be the main ingredient in the proof of the main result of this article, Main Theorem 14.5. As is explained there, the arithmetic formula expresses — in some sense — the special derivative of the orbit equation 14.2. We mention also a rather well-known geometric analogue (Theorem 13.2, 1.) which expresses the special value of the orbit equation. Recall the SO-equivariant line bundle $\overline{\mathcal{E}} = (\mathcal{E}, h_{\mathcal{E}})$ with Hermitian metric on $\mathbb{D}_{\mathbf{O}}$. We denote $E = \mathcal{E}_{\mathbb{C}}$ (cf. 11.7).

Theorem 13.2. *Assume 6.6.*

Under the conditions above, we have:

1. *(geometric formula)*

$$\begin{aligned} & \sum_q \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) \text{vol}_E(\text{M}(\Delta^K \mathbf{O}(L))) \\ &= \sum_q \text{vol}_E(\text{Z}(L_{\mathbb{Z}}, < -q >, a_q; K)) \end{aligned}$$

2. *(arithmetic formula)*

$$\begin{aligned} & \sum_q \tilde{\mu}'(L_{\mathbb{Z}}, < -q >, a_q; 0) \text{vol}_E(\text{M}(\Delta^K \mathbf{O}(L))) \\ &+ \sum_q \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{M}(\Delta^K \mathbf{O}(L))) \\ &= \sum_q \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{Z}(L_{\mathbb{Z}}, < -q >, a_q; K)) \end{aligned}$$

in $\mathbb{R}^{(p)}$.

Proof. Let $f_0 := \Psi(F)$ be the Borcherds lift of F — cf. section 11.3 ff. and especially Theorem 11.14.

We sketch the proof of 1. The assumption on m and the Witt rank imply that the Eisenstein series associated with $L_{\mathbb{Z}}$ (see [17, Section 4]) is holomorphic, i.e. there is a

$$G \in \text{HolModForm}(\text{SL}_2'(\mathbb{Z}), \text{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^*, \frac{m}{2})$$

with $a_n(G)(\varphi) = -\tilde{\mu}(L_{\mathbb{Z}}, < n >, \varphi; 0)$ for $n > 0, n \in \mathbb{Q}$ and $a_0(G)(\varphi) = \varphi(0)$. Hence relation (19) yields:

$$\sum_q \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) = a_0(0). \quad (22)$$

In addition, we have the relation

$$\begin{aligned} a_0(0) \int_M c_1(\Xi^* E, \Xi^* h)^m &= 2 \int_M d \, d^c \log h(f_0) \wedge c_1(\Xi^* E, \Xi^* h)^{m-1} \\ &= 2 \int_{\operatorname{div}(f_0) \cap M} c_1(\Xi^* E, \Xi^* h)^{m-1}. \end{aligned} \quad (23)$$

The justification that here everything is integrable and that the second equality is true without contributions from the boundary, can be found in [6]. From (22) and (23), 1. follows, taking the description of the divisor of f_0 (Theorem 11.14) into account.

2. We may calculate $\widehat{\operatorname{vol}}_{\mathcal{E}, p}(\operatorname{M}(\Delta^K \mathbf{O}(L)))$ in the following way:

First assume w.l.o.g. (e.g. by taking a lattice with large discriminant in the construction of F or by just pulling back $\Psi(F)$ afterwards) that K is neat.

We know by 11.16 that $\Xi^*(\mathcal{E})$ is ample on $\operatorname{M}(\Delta^K \mathbf{O})$ and that some power of it has no base points on $\operatorname{M}(\Delta^K \mathbf{O})$. Choose some sufficiently fine, smooth w.r.t. K , complete, and projective Δ (by Lemma 10.23 the arithmetic volume does not depend on this choice) such that all models exist and all special cycles involved embed closely as projective schemes (cf. 9.6). $\Xi^*(\mathcal{E})^{\otimes k}$, for some positive k , defines a morphism

$$\alpha : \operatorname{M}(\Delta^K \mathbf{O}) \rightarrow \mathbb{P}^r(\mathbb{Z}_{(p)}),$$

whose restriction to $\operatorname{M}(\Delta^K \mathbf{O})$ is an embedding. Let D be the boundary divisor. By Main Theorem 6.9, M and D are defined over $\mathbb{Z}_{(p)}$. We have $\dim(\alpha(D)_{\mathbb{C}}) = \text{Witt rank of } L_{\mathbb{Q}} - 1$, hence $\dim(\alpha(D)) = \text{Witt rank of } L_{\mathbb{Q}}$, because $\Xi^*(\mathcal{E})_{\mathbb{C}}^{\otimes k}$ induces the Baily-Borel compactification. In addition, we may also choose k and F such that $f_0 \in H^0(\operatorname{M}(\Delta^K \mathbf{O}), \Xi^*(\mathcal{E})_{\text{horz}}^{\otimes k})$ (Theorem 11.14). Up to increasing k again, we may find hyperplanes $H_1, \dots, H_{m-2} \subset \mathbb{P}^r(\mathbb{Z}_{(p)})$ which intersect properly with $\alpha(\operatorname{div}(f_0))$, and we may avoid that they intersect simultaneously in any closed subset Z of dimension $< m - 2$.

Our assumptions on m and the Witt rank imply that we can take Z to be the union of $\alpha(D)$ ($\dim = \text{Witt rank} < m - 2$) and the locus, where $\alpha_{\mathbb{F}_p}$ is not finite ($\dim < m - 2$).

We may clearly assume that already $H_{1, \mathbb{C}}$ and $H_{2, \mathbb{C}}$ do not intersect in $\alpha(D)_{\mathbb{C}}$.

Let f_1, \dots, f_{m-2} be the corresponding sections.

The construction assures that also $\operatorname{div}(f_0), \operatorname{div}(f_1), \dots, \operatorname{div}(f_{m-2})$ intersect properly and not in D simultaneously. Therefore by Theorem 11.14, 2.,

$$\operatorname{div}(f_0) \cdot \operatorname{div}(f_1) \cdots \operatorname{div}(f_{m-2}) = \frac{1}{2} \sum_{q < 0} Z(L_{\mathbb{Z}}, < -q >, a_q; K) \cdot \operatorname{div}(f_1) \cdots \operatorname{div}(f_{m-2}).$$

Note that the arithmetic volume is the sum of this expression and

$$\frac{1}{2} \int g_0 * g_1 * \cdots * g_{m-2}, \quad (24)$$

where

$$g_i = -\log \Xi^* h(f_i)$$

(and the star product has to be interpreted in the sense of 10.8). In the following we will denote

$$\Omega := c_1(\Xi^*(\mathcal{E}), \Xi^*(h_{\mathcal{E}}))$$

to be the first Chern form of the bundle $\Xi^*(\mathcal{E})$ with respect to the (log-singular) Hermitian metric $\Xi^*h_{\mathcal{E}}$.

On any parametrization defined by a point-like boundary component as in 5.5 it is given by

$$-d d^c \log(Q_{z'}(Y))$$

where $Z = X + iY \in U_{\mathbb{C}} \cong \mathbb{D}_{\mathbf{H}_0}[U, 0]$.

By Theorem 13.3 below, we may write (24) as

$$\begin{aligned} &= \frac{1}{2} \int_{M(\frac{K}{\Delta} \mathbf{O})_{\mathbb{C}}} g_0 k \Omega \wedge \cdots \wedge k \Omega \\ &+ \frac{1}{2} \int_{\text{div}(f_0) \cap M(\frac{K}{\Delta} \mathbf{O})_{\mathbb{C}}} g_1 * \cdots * g_{m-2}, \end{aligned}$$

hence we get the equation ($\in \mathbb{R}^{(p)}$)

$$\begin{aligned} &k^{m-1} \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(M(\frac{K}{\Delta} \mathbf{O}(L))) \\ &= \frac{1}{2} k^{m-2} \int_{M(\frac{K}{\Delta} \mathbf{O})_{\mathbb{C}}} g_0 \Omega \wedge \cdots \wedge \Omega \\ &+ \frac{1}{2} k^{m-2} \sum_q \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(Z(L_{\mathbb{Z}}, < -q >, a_q; K)), \end{aligned}$$

(cf. also 10.16) and therefore the required one, taking into account that

$$\frac{1}{2} \sum_q \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) = k = \frac{a_0(0)}{2}$$

(geometric formula — part 1.) and by Theorem 11.14, 3.:

$$\begin{aligned} &\int_{M(\frac{K}{\Delta} \mathbf{O})_{\mathbb{C}}} g_0 \Omega \wedge \cdots \wedge \Omega \equiv \\ &- \text{vol}_{\mathcal{E}}(M(\frac{K}{\Delta} \mathbf{O})_{\mathbb{C}}) \sum_q \tilde{\mu}'(L_{\mathbb{Z}}, < -q >, a_q; 0) \end{aligned}$$

in $\mathbb{R}^{(p)}$. □

Theorem 13.3. *Let $L_{\mathbb{Q}}$ be as before, i.e. of signature $(m-2, 2)$ with $m \geq 5$, or $m = 4$ and $L_{\mathbb{Q}}$ of Witt rank 1.*

Let f_0, \dots, f_{m-2} be sections of $\Xi^(\mathcal{E})^{\otimes k}$ on $M(\frac{K}{\Delta} \mathbf{O})$, such that $\text{div}(f_0), \dots, \text{div}(f_{m-2})$ intersect properly:*

$$\bigcap_i \text{supp}(\text{div}(f_i)) = \emptyset,$$

such that

$$\text{supp}(\text{div}(f_1)) \cap \text{supp}(\text{div}(f_2)) \cap \text{supp}(D) = \emptyset$$

(Witt rank 2), resp.

$$\text{supp}(\text{div}(f_1)) \cap \text{supp}(D) = \emptyset$$

(Witt rank 1).

Then we have

$$\begin{aligned} & \int_{M(\frac{K}{\Delta}\mathbf{O})_{\mathbb{C}}} g_0 * g_1 * \cdots * g_{m-2} \\ &= \int_{M(K\mathbf{O})_{\mathbb{C}}} g_0 k\Omega \wedge \cdots \wedge k\Omega \\ &+ \int_{\text{div}(f_0) \cap M(K\mathbf{O})_{\mathbb{C}}} g_1 * \cdots * g_{m-2}. \end{aligned}$$

All occurring integrals exist.

Proof. According to [8, Theorem 1.14], we have

$$\begin{aligned} & \int_{M(\frac{K}{\Delta}\mathbf{O})_{\mathbb{C}}} g_0 * \cdots * g_{m-2} \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{M(\frac{K}{\Delta}\mathbf{O})_{\mathbb{C}} - B_{\varepsilon}(D)} g_0(k\Omega) \wedge \cdots \wedge (k\Omega) \right. \\ &\quad \left. - \int_{\partial B_{\varepsilon}(D)} g \wedge d^c g_0 - g_0 \wedge d^c g \right) \\ &+ \int_{\text{div}(f_0) \cap M(K\mathbf{O})_{\mathbb{C}}} g, \end{aligned}$$

where

$$g = g_1 * \cdots * g_{m-2}.$$

The integral

$$\int_{M(K\mathbf{O})} g_0(k\Omega) \wedge \cdots \wedge (k\Omega)$$

exists by [6, Theorem 2] because we excluded the cases $m = 3$, Witt rank 1 (modular curves) and $m = 4$, Witt rank 2 (product of modular curves).

13.4. We will need special neighborhoods of points on the boundary divisor of $M(\frac{K}{\Delta}\mathbf{O})$. By Main Theorem 6.9, 3. any such point lies on a stratum

$$[\text{Stab}_{\Gamma([\sigma])} \setminus M^{(K_{[\sigma]})}\mathbf{B}_{[\sigma]}]_{\mathbb{C}},$$

where \mathbf{B} corresponds (5.5) to an isotropic line or an isotropic surface in $L_{\mathbb{Q}}$. (The corresponding boundary stratum in the Baily-Borel compactification is 0, resp. 1-dimensional.) We will prepare special neighborhoods of these points

and call them of **first (resp. second) type** for the rest of this section. By the boundary isomorphism 6.9, 4., we transfer the neighborhood to the boundary of the mixed Shimura variety (compactified only along the unipotent fibre):

$$M_{(\Delta')^0}^{(K')}\mathbf{B}_{\mathbb{C}},$$

which is a torus embedding constructed from the (relative) torus $M^{(K')}\mathbf{B}_{\mathbb{C}}$ over $M^{(K'')}\mathbf{B}/U_{\mathbf{B}}\mathbb{C}$ (6.8).

13.5. For a point of the first type, we have

$$\mathbf{B} \cong \mathbf{H}_0[0, U_{\mathbb{Z}(p)}], \quad (25)$$

where $U_{\mathbb{Z}(p)} = I \otimes (I^{\perp}/I)$ (5.5) and the trivialization s of $\Xi^*\mathcal{E}$ described in 8.2. It is of the form

$$s(z_1, \dots, z_{m-2}) = (\log(z_1)\lambda_1 + \dots + \log(z_n)\lambda_n + \psi(z_{k+1}, \dots, z_{m-2}))z'$$

for $z_1, \dots, z_{m-2} \in B_{\epsilon}(0)^{m-2}$, z' is a vector in the fibre of \mathcal{E} above the point corresponding to the splitting (25), i.e. $z' \in I^*$. The λ_i span the rational polyhedral cone σ corresponding to our boundary point and form part of a basis of $(U_{\mathbb{Q}} \cap K')(-1)$ (Δ is smooth w.r.t. K), and ψ is a holomorphic embedding $B_{\epsilon}(0)^{m-2-k} \rightarrow U_{\mathbf{B}}(\mathbb{C}) = U_{\mathbb{C}}$.

Let $r_i := |z_i|$. The norm of s is given by

$$\sum_{i \leq n, j \leq n} \langle \lambda_i, \lambda_j \rangle_{z'} \log(r_i) \log(r_j) + \sum_{i \leq n} \log(r_i) \Psi_i(z_1, \dots, z_{m-2}) + \Psi_0(z_1, \dots, z_{m-2}),$$

where the $\Psi_i, i > 0$ are smooth functions on $B_{\epsilon}(0)^{m-2}$ satisfying $\mathrm{d} \mathrm{d}^c \Psi_i = 0$, and Ψ_0 is smooth.

13.6. For a point of the second type, we have

$$\mathbf{B} \cong \mathbf{H}_1(I)[\Lambda^2 I, I \otimes (I^{\perp}/I)], \quad (26)$$

(5.5) and the trivialization s of $\Xi^*\mathcal{E}$ described in 8.2. It is of the form

$$s(z_1, z_2, \dots, z_{m-3}, z_{m-2}) \mapsto (\lambda \log(z_1)) \circ (\psi_2(z_2, \dots, z_{m-3}))(\psi_1(z_{m-2})).$$

Here ψ_1 is a local trivialization of $\mathcal{E}_{\mathbb{C}}$ over a small neighborhood over the image of $\mathbb{D}_{\mathbf{H}_1(I)}$ in $\mathbb{D}_{\mathbf{H}_1(I)[\Lambda^2 I, I \otimes (I^{\perp}/I)]}$ such that $p(\psi_1(0))$ is a representative of the projection of the boundary point. Since \mathcal{E} is the tautological bundle, we may express this by saying that ψ_1 maps to I^* inducing an embedding of $B_{\epsilon}(0)$ into $\mathbb{P}(I^*)$. ψ_2 is a holomorphic map $B_{\epsilon}(0)^{m-4} \rightarrow W_{\mathbf{B}}(\mathbb{C})$ which is modulo $U_{\mathbf{B}}(\mathbb{C})$ an open embedding into $V_{\mathbf{B}}(\mathbb{C})/\mathrm{Stab}(x)$ for all x in the image of $p(\psi_1(z_{m-2}))$. λ is a basis of $(U_{\mathbf{B}}(\mathbb{Q}) \cap K')(-1)$ and corresponds to the rational polyhedral cone σ (here there is only one possible).

Its norm is of the form

$$h_{\mathcal{E}}(s) = \log(r_1)\Psi_1(z_{m-2}) + \Psi_0(z_2, \dots, z_{m-3}, z_{m-2}),$$

where $\mathrm{d} \mathrm{d}^c \Psi_1 = 0$, and Ψ_0 is smooth. Note that Ψ_1 does only depend on z_{m-2} .

13.7. We will prepare an ε -tube neighborhood of D as follows. We prepared neighborhoods around every point of the boundary above. Take a finite cover $\{U_i\}$ consisting of these. Let τ_i be a partition of unity defined on $M(\frac{K}{\Delta}\mathbf{O}(L))_{\mathbb{C}}$ for the chosen cover, consisting of the neighborhoods considered in 13.5 and 13.6. Consider a neighborhood U_i with coordinates $z_{i,1}, \dots, z_{i,m-2}$. D has the equation $(z_{i,1})^{a_1} \dots (z_{i,m-2})^{a_{m-2}} = 0$. On intersections $U_i \cap U_j$ we have

$$z_{j,k_j} = z_{i,k}(f_{i,k,0}^j(z_{i,1}, \dots, z_{i,m-2}) + z_{i,k}f_{i,k}^j(z_{i,1}, \dots, z_{i,m-2})),$$

where $f_{i,k,0}^j(z_{i,1}, \dots, z_{i,m-2})$ does not depend on $z_{i,k}$ and is everywhere non-zero on the overlap in question and where k_j is the index of the k -th divisor component in the neighborhood U_j . We define

$$r'_{i,k} = \sum_j \tau_j |z_j^{j,k_j}| \quad \text{on } U_i.$$

If the k -komponent does not exist in U_j , we omit the term in the sum. Define $r' = \min_{k|a_k \neq 0} r'_{i,k}$. This does not depend on i anymore and is a well-defined function in a neighborhood of the divisor. A global ε -tube neighborhood around D may therefore be described as

$$B_\varepsilon(D) := \{z \in M(\frac{K}{\Delta}\mathbf{O}(L))_{\mathbb{C}} \mid r'(z) \leq \varepsilon\}.$$

In a local neighborhood U_i write

$$r'_{i,k} = r_{i,k} \left(\sum_j \tau_j |f_{0,i,k}^j(z_{i,1}, \dots, z_{i,m-2})| + r_{i,k}g \right),$$

where g is a bounded C^∞ function. If ε is small enough, $0 < r'_i \leq \varepsilon$ will ensure $r_{i,k} = |z_{i,k}| > 0$.

On U_i we may cover $\partial B_\varepsilon(D)$ by (we omit the index i because it will be fixed from now on)

$$St'_k(\varepsilon) = \{z \in U_i \mid r'_k = \varepsilon, r'_j \geq \varepsilon \text{ if } a_j \neq 0\},$$

for all k , where $a_k \neq 0$. Note that on overlaps the $r'_{i,k}$ and $r'_{j,k}$ are the same up to permutation of the indices k .

We have for small r'_k and some bounded C^∞ -function h :

$$\begin{aligned} r_k &= r'_k \left(\left(\sum \tau_j |f_{0,k}^j(z_1, \dots, z_{m-2})| \right)^{-1} + r'_k h \right), \\ dr_k &= dr'_k \left(\sum \tau_j |f_{0,k}^j(z_1, \dots, z_{m-2})| \right)^{-1} \\ &\quad + r'_k dr'_k h \\ &\quad + r'_k d \left(\left(\sum \tau_j |f_{0,k}^j(z_1, \dots, z_{m-2})| \right)^{-1} + r'_k h \right). \end{aligned} \tag{27}$$

13.8. In any of the neighborhoods U_i , we will write the quantities in question in the basis

$$d\varphi_1, \dots, d\varphi_n, dr_1, \dots, dr_n, dr'_k.$$

Only the term

$$\omega := d\varphi_1 \wedge \dots \wedge d\varphi_n \wedge dr_1 \wedge \dots \wedge dr_n$$

gives a non-zero contribution in integrals over $St'_k(\varepsilon)$. In what follows, we understand:

$$|f(z)\omega| := |f(z)|\omega.$$

We number the coordinates in such a way that D is given by the equation $z_1 \dots z_n = 0$ on U_i .

Recall the notion of log-log-growth of infinite order [9, 2.2]. A **log-log-form** [9, 2.22] is a differential form generated (over smooth forms) by functions of log-log-growth of infinite order and the differentials

$$\frac{d\varphi_1}{\log(r_1)}, \dots, \frac{d\varphi_n}{\log(r_n)}, \frac{dr_1}{r_1 \log(r_1)}, \dots, \frac{dr_n}{r_n \log(r_n)}$$

such that d, d^c and dd^c of it are again of this type.

Call a form a **(*)-form** for an index k , if it is generated by functions of log-log-growth of infinite order and the differentials

$$d\varphi_1, \dots, d\varphi_n, \frac{dr_1}{r_1 \log(r_1)}, \dots, \frac{dr_n}{r_n \log(r_n)}$$

and $f dr'_k$ for any f , which is smooth outside D .

Lemma 13.9. *For every (*)-form σ for the index k :*

$$\int_{St'_k(\varepsilon)} |\sigma| = O(\log(|\log \varepsilon|)^M)$$

for some M .

Proof. It suffices to show that on each of the sets

$$St'_k(\varepsilon, R) = \{z \in B(R) \mid r'_k = \varepsilon, r_j \geq \delta\varepsilon \text{ if } a_j \neq 0\},$$

for some small δ (note that we wrote r_j instead of r'_j), we have the estimate:

$$\int_{St'_k(\varepsilon, R)} |\sigma| = O(\log(|\log \varepsilon|)^M).$$

This is easy, see e.g. [6]. The (maybe singular) terms $f dr'_k$ play no role because terms involving dr'_k do not give any contribution to the integral. \square

Corollary 13.10. *For every log-log growth form ξ :*

$$\lim_{\varepsilon \rightarrow 0} \int_{St'_k(\varepsilon)} |\xi| = 0.$$

Proof. We have by (27)

$$\frac{1}{\log(r_k)} d\varphi_i \prec \frac{1}{\log(r'_k)} \sigma,$$

and

$$\frac{dr_k}{r_k \log(r_k)} \prec \frac{1}{\log(r'_k)} \sigma,$$

where the σ are (*)-forms w.r.t. k .

If ξ has correct degree, by definition, it involves at least either $\frac{1}{\log(r_k)} d\varphi_k$ or $\frac{dr_k}{r_k \log(r_k)}$. Hence

$$\xi \prec \frac{1}{\log(r'_k)} \sigma,$$

for some (*)-form σ . The statement follows using 13.9 because, of course, \log grows faster than any power of $\log\text{-}\log$. \square

13.11. First of all $\text{supp}(\text{div}(f_1)) \cap \text{supp}(\text{div}(f_2)) \cap \text{supp}(D) = \emptyset$ by assumption, hence we may represent (cf. 10.9):

$$g = \sigma g_{1,2} \Omega^{n-2} + d d^c((1 - \sigma) g_{1,2}) g',$$

where σ is equal to 1 in a neighborhood of D and $g_{1,2} = g_1 * g_2$. We may represent this by

$$g_{1,2} = \sigma_1 g_1 \Omega + d d^c(\sigma_2 g_1) g_2,$$

where $\sigma_1 + \sigma_2 = 1$ is a partition of unity of the following form: The intersection of $\text{div}(f_1)$ with D occurs precisely on the pre-image under the projection on the Baily-Borel compactification of a set of isolated points in the 1-dimensional boundary stratum isomorphic to some $M(K\mathbf{H}(I))$. Choose a C^∞ -function $p\sigma$ on $M(K\mathbf{H}(I))$ with support on some disc around one of these points, which is 1 on a smaller neighborhood of the point.

We may assume that in $B_\varepsilon(D)$ for small ε , σ_1 is just the pre-image of $p\sigma$ under the projection $M_{(\Delta')^\circ}^{(K')}\mathbf{B} \rightarrow M(K\mathbf{H}(I))$.

We have hence in a neighborhood of D :

$$\begin{aligned} g &= \sigma_1 g_1 \Omega^{n-1} + \sigma_2 g_2 \Omega^{n-1} \\ &\quad + ((d\sigma_2 \wedge d^c g_1) g_2 - (d^c \sigma_2 \wedge d g_1) g_2 + (d d^c \sigma_2) g_1 g_2) \wedge \Omega^{n-2}. \end{aligned} \tag{28}$$

In any of the neighborhoods considered in 13.5 and 13.6 we have

$$g_0 = \sum_{j \leq n} a_j \log(r_j) + \log(h_{\mathcal{E}}(s)) + \psi(z) \tag{29}$$

for some harmonic function ψ , where s is the corresponding trivializing section of $\Xi^*(\mathcal{E})$.

Recall that we want to show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(D)} g \wedge d^c g_0 - g_0 \wedge d^c g = 0.$$

Inserting (28) and (29) into this, since in any case (by 13.10) the limit of integrals of log-log forms is 0, we are reduced to show, on the one hand (first line in 28), that

$$\lim_{\varepsilon \rightarrow 0} \int_{St'_k(\varepsilon)} |d\varphi_j \sigma_l g_l \Omega^{m-3}| = 0$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{St'_k(\varepsilon)} |\log(r_j) d^c(\sigma_l g_l) \Omega^{m-3}| = 0,$$

where $l = 1, 2$, in neighborhoods of both types. Every other term in (29) inserted into

$$\lim_{\varepsilon \rightarrow 0} \int_{St'_k(\varepsilon)} |g \wedge d^c g_0 - g_0 \wedge d^c g|$$

yields a limit over an integral of a form of log-log type, which is zero by Lemma 13.10.

If $j = k$ we may rewrite r_j by means of r'_k and apply (32) of Lemma 13.12. In the other case we apply (33) of Lemma 13.12. (Note that $d\varphi_j$ is of the form $\log(r_j)$ times a log-log-form.) After this, the vanishing of the limit $\varepsilon \rightarrow 0$ follows from Lemma 13.9.

On the other hand (second line in 28), we have to show

$$\lim_{\varepsilon \rightarrow 0} \int_{St'_1(\varepsilon)} |d\varphi_1 (d\sigma_2 \wedge d^c g_1) - d^c \sigma_2 \wedge d g_1 + (d d^c \sigma_2) g_1) \wedge g_2 \wedge \Omega^{m-4}| = 0 \quad (30)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{St'_1(\varepsilon)} |\log(r_1) (d\sigma_2 \wedge d^c g_1) - d^c \sigma_2 \wedge d g_1 + (d d^c \sigma_2) g_1) \wedge d^c g_2 \wedge \Omega^{m-4}| = 0 \quad (31)$$

Here $j = k = 1$ because this is non-zero only in neighborhoods of the second type, and there cannot be an intersection of components above $M(K\mathbf{H}(I))$.

The estimate (34) of Lemma 13.12, we get vanishing of the limit $\varepsilon \rightarrow 0$ of (30) by Lemma 13.9 again.

For (31) note that we constructed σ_2 (in $B_\varepsilon(D)$!) as the pre-image (under the projection $M(\Delta_1^{K_1}\mathbf{B}) \rightarrow M(K\mathbf{H}(I))$) of a function supported on a small disc around the zeros or poles of f_1 in the boundary of the Baily-Borel compactification. Let X be the pre-image of these discs in some neighborhood of the boundary in $M(\Delta_1^K\mathbf{B})$. Hence the form

$$d(g_0(d\sigma_2 \wedge d^c g_1 - d^c \sigma_2 \wedge d g_1 + (d d^c \sigma_2) g_1) \wedge d^c g_2 \wedge d^c \log(h_\varepsilon(s)) \wedge \Omega^{m-3})$$

is supported only on X . Applying the Theorem of Stokes to this function on $\partial B_\varepsilon(D)$, we get

$$\begin{aligned}
0 = & - \int_{\partial B_\varepsilon(D)} g_0 (\mathrm{d} \sigma_2 \wedge \mathrm{d}^c g_1 - \mathrm{d}^c \sigma_2 \wedge \mathrm{d} g_1 + (\mathrm{d} \mathrm{d}^c \sigma_2) g_1) \\
& \wedge \mathrm{d}^c g_2 \wedge \Omega \wedge \Omega^{m-5} \\
& + \int_{\partial B_\varepsilon(D)} g_0 (\mathrm{d} \sigma_2 \wedge \mathrm{d}^c g_1 - \mathrm{d}^c \sigma_2 \wedge \mathrm{d} g_1 + (\mathrm{d} \mathrm{d}^c \sigma_2) g_1) \\
& \wedge \mathrm{d} \mathrm{d}^c g_2 \wedge \mathrm{d}^c \log(h_\mathcal{E}(s)) \wedge \Omega^{m-5} \\
& + \int_{\partial B_\varepsilon(D)} g_0 \left(-\mathrm{d} \sigma_2 \wedge \mathrm{d} \mathrm{d}^c g_1 - \underbrace{\mathrm{d} \mathrm{d}^c \sigma_2 \wedge \mathrm{d} g_1 + \mathrm{d} \mathrm{d}^c \sigma_2 \wedge \mathrm{d} g_1}_{=0} \right) \\
& \wedge \mathrm{d}^c g_2 \wedge \mathrm{d}^c \log(h_\mathcal{E}(s)) \wedge \Omega^{m-5} \\
& + \int_{\partial B_\varepsilon(D)} \mathrm{d} g_0 \wedge (\mathrm{d} \sigma_2 \wedge \mathrm{d}^c g_1 - \mathrm{d}^c \sigma_2 \wedge \mathrm{d} g_1 + (\mathrm{d} \mathrm{d}^c \sigma_2) g_1) \\
& \wedge \mathrm{d}^c g_2 \wedge \mathrm{d}^c \log(h_\mathcal{E}(s)) \wedge \Omega^{m-5}
\end{aligned}$$

The limit of the first line is the one we have to estimate. We may hence instead consider the remaining three lines. Those we may again restrict to small neighborhoods U_i (all of the second type!) and estimate the absolute value of the integral. Here all problematic terms are of the form

$$\int_{St'_1(\varepsilon)} |\log(r_1) \xi' \wedge \xi \wedge \mathrm{d}^c \log(h_\mathcal{E}(s)) \wedge \Omega^{m-4}|$$

or

$$\int_{St'_1(\varepsilon)} \left| \frac{\mathrm{d} r_1}{r_1} \wedge \xi' \wedge \xi \wedge \mathrm{d}^c g_2 \wedge \mathrm{d}^c \log(h_\mathcal{E}(s)) \wedge \Omega^{m-5} \right|,$$

where ξ' is a smooth form of degree at least 1, generated by $r_{m-2} \mathrm{d} \varphi_{m-2}$ and $\mathrm{d} r_{m-2}$ and ξ is log-log. The latter poses no problem because $\frac{\mathrm{d} r_1}{r_1}$ is, up to terms involving $\mathrm{d} r'_1$, a smooth form (27).

Now, by the estimate (35) of Lemma 13.12, we get vanishing of the limit $\varepsilon \rightarrow 0$ by Lemma 13.9 again. \square

Lemma 13.12. *Let U be a neighborhood of any type and let i be the Witt rank of L . Then for any log-log form ξ of rank $2(m-2-i)-1$, we have*

$$\xi \wedge \Omega^i \prec \frac{1}{\log(r'_k)^2} \sigma, \tag{32}$$

where σ is a $(*)$ -form w.r.t. to k .
Also we have for any $j \neq k$

$$\xi \wedge \Omega^i \prec \frac{1}{\log(r'_k) \log(r_j)} \sigma, \tag{33}$$

where σ is another $(*)$ -form w.r.t. k .

If U is a neighborhood of the second type (13.6), for any log-log form $\xi' \wedge \xi$ of degree $2(m-2)-5$, where ξ' is a smooth form of degree at least 1, involving only $r_{m-2} d\varphi_{m-2}$ and dr_{m-2} , where r_{m-2}, φ_{m-2} are the polar coordinates of the projection to a $M^K(\mathbf{H}(I))$ in a neighborhood of the point, we have

$$d\varphi_1 \wedge \xi' \wedge \xi \wedge \Omega \prec \frac{1}{\log(r'_1)} \sigma, \quad (34)$$

and

$$\xi' \wedge \xi \wedge d^c \log(h_{\mathcal{E}}(s)) \wedge \Omega \prec \frac{1}{\log(r'_1)^2} \sigma, \quad (35)$$

where the σ are $(*)$ -forms w.r.t. the index 1.

Proof. We have in any case:

$$\begin{aligned} d^c \log(r_i) &= d\varphi_i, \\ d \log(r_i) &= \frac{dr_i}{r_i}, \end{aligned}$$

and

$$\begin{aligned} d^c \log h_{\mathcal{E}}(s) &= \frac{d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}, \\ d \log h_{\mathcal{E}}(s) &= \frac{d h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}, \\ \Omega \sim d d^c \log h_{\mathcal{E}}(s) &= \frac{d h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \frac{d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} + \frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \end{aligned}$$

In a neighborhood of the first type (13.5), we may write

$$h_{\mathcal{E}}(s) = \sum_{i,j \leq n} \langle \lambda_i, \lambda_j \rangle \log(r_i) \log(r_j) + \sum_{i \leq n} \log(r_i) \psi_i(z) + \psi_0(z),$$

where ψ_i are harmonic functions and ψ_0 is smooth. Here the λ_i are linearly independent and $\langle \lambda_i, \lambda_j \rangle > 0$ if $i \neq j$. Hence for $i \neq j$ always

$$h_{\mathcal{E}}(s) \gg \log(r_i) \log(r_j), \quad (36)$$

and if $Q_L(\lambda_i) > 0$

$$h_{\mathcal{E}}(s) \gg \log(r_i)^2. \quad (37)$$

Hence

$$\begin{aligned} d^c h_{\mathcal{E}}(s) &= \log(r_k) \left(\sum_{j \leq n} \langle \lambda_k, \lambda_j \rangle d\varphi_j + d^c \psi_k \right) + \sum_{i \leq n} d\varphi_i \psi_i \\ &+ \sum_{i \leq n, j \leq n, i \neq k} \langle \lambda_i, \lambda_j \rangle \log(r_i) d\varphi_j + \sum_{i \leq n, i \neq k} \log(r_i) d^c \psi_k + d^c \psi_0 \end{aligned} \quad (38)$$

$$\begin{aligned}
d h_{\mathcal{E}}(s) &= \log(r_k) \left(\sum_{j \leq n} \langle \lambda_k, \lambda_j \rangle \frac{dr_j}{r_j} + d \psi_k \right) + \sum_{i \leq n} \frac{dr_j}{r_j} \psi_i \\
&\quad + \sum_{i \leq n, j \leq n, i \neq k} \langle \lambda_i, \lambda_j \rangle \log(r_i) \frac{dr_j}{r_j} + \sum_{i \leq n, i \neq k} \log(r_i) d \psi_k + d \psi_0 \quad (39) \\
d d^c h_{\mathcal{E}}(s) &= d \varphi_k \left(\sum_{j \leq n} \langle \lambda_k, \lambda_j \rangle \frac{dr_j}{r_j} + d \psi_k \right) + \sum_{i \leq n} \frac{dr_j}{r_j} d^c \psi_i \\
&\quad + \sum_{i \leq n, j \leq n, i \neq k} \langle \lambda_i, \lambda_j \rangle d \varphi_i \frac{dr_j}{r_j} + \sum_{i \leq n, i \neq k} d \varphi_i d \psi_k + d d^c \psi_0. \quad (40)
\end{aligned}$$

We now substitute $\frac{dr_k}{r_k}$ by a form of shape

$$f \frac{dr'_k}{r'_k} + \xi,$$

where ξ is smooth. This is possible by means of formula (27).
First assume $i = 2$. Ω^2 is proportional to

$$\frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \wedge \frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \quad (41)$$

$$+ \frac{d h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \frac{d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \wedge \frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \quad (42)$$

In (41), multiplying out the expression (40) squared, we have for every occurring summand S

$$S \prec \frac{1}{\log(r_k)^2} \sigma \prec \frac{1}{\log(r'_k)^2} \sigma,$$

where σ is a $(*)$ -form, using the estimate (36).

In (42), multiplying out the product of (38–40), we have

$$S \prec \frac{1}{\log(r_k)^2} \sigma \prec \frac{1}{\log(r'_k)^2} \sigma,$$

using the estimate (36) again, except for the summands of the form

$$\log(r_k) \log(r_k) \left(\sum_{j \leq n} \langle \lambda_k, \lambda_j \rangle d \varphi_j + d^c \psi_k \right) \wedge \left(\sum_{j \leq n} \langle \lambda_k, \lambda_j \rangle \frac{dr_j}{r_j} + d \psi_k \right) \wedge S, \quad (43)$$

where S is any summand of (40).

But now in the expression $\xi \wedge \Omega^2$ either occurs a $\frac{d \varphi_k}{\log(r_k)}$ from ξ , hence the estimate (32) is true, or a $r_k d \varphi_k$ occurs in S which satisfies a much stronger estimate, or a $d \varphi_k$ occurs in S . It occurs, however, multiplied with $(\sum_{j \leq n} \langle \lambda_k, \lambda_j \rangle \frac{dr_j}{r_j} + d \psi_k)$ so (43) is zero in that case.

The estimate (33) is more easy and left to the reader.

Now assume $i = 1$. Then, by the estimate (37), already every summand S in (38–40) divided by $h_{\mathcal{E}}(s)$ satisfies

$$S \prec \frac{1}{\log(r_k)} \sigma \prec \frac{1}{\log(r'_k)} \sigma,$$

with a $(*)$ -form σ .

The estimates (34, 35) do not involve neighborhoods of the first type. In a neighborhood *of the second type* (13.6), we may write

$$h_{\mathcal{E}}(s) = \log(r_1) \psi_1(z_{m-2}) + \psi_0(z),$$

where ψ_1 is harmonic and depends only on z_{m-2} and ψ_0 is smooth. Clearly in this case:

$$h_{\mathcal{E}}(s) \gg \log(r_1). \quad (44)$$

Also

$$\begin{aligned} d^c h_{\mathcal{E}}(s) &= \log(r_1) d^c \psi_1(z_{m-2}) + d\varphi_1 \psi_1(z_{m-2}) + d^c \psi_0(z), \\ d h_{\mathcal{E}}(s) &= \log(r_1) d \psi_1(z_{m-2}) + \frac{dr_1}{r_1} \psi_1(z_{m-2}) + d \psi_0(z), \\ d d^c h_{\mathcal{E}}(s) &= d\varphi_1 d \psi_1(z_{m-2}) + \frac{dr_1}{r_1} d^c \psi_1(z_{m-2}) + d d^c \psi_0(z). \end{aligned}$$

We now substitute again $\frac{dr_1}{r_1}$ by a form of shape

$$f \frac{dr'_1}{r'_1} + \xi,$$

where ξ is smooth. This is possible by means of formula (27).

Here, for (32), we may argue exactly as before. (33) is vacuous in this type of neighborhood. For (34): Write once again Ω as

$$\frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \quad (45)$$

$$+ \frac{d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \wedge \frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \quad (46)$$

In (45), using (44), we have for any summand

$$S \prec \frac{1}{\log(r'_1)} \sigma,$$

for a $(*)$ -form σ .

In (46), using (44) again, we get the same for any summand except possibly for

$$\frac{1}{h_{\mathcal{E}}(s)^2} \log(r_1)^2 d^c \psi_1(z_{m-2}) \wedge d \psi_1(z_{m-2}),$$

which cancels, however, with ξ' because $d^c \psi_1(z_{m-2}) \wedge d \psi_1(z_{m-2})$ is proportional to $r_{m-2} d \varphi_{m-2} \wedge d r_{m-2}$.

For (35): Consider:

$$d^c \log(h_{\mathcal{E}}(s)) \wedge \Omega = \frac{d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \wedge \frac{d d^c h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \quad (47)$$

Using (44) again, we now have for any summand in

$$S \prec \frac{1}{\log(r'_1)^2} \sigma,$$

for a $(*)$ -form σ , except possibly for

$$\frac{1}{h_{\mathcal{E}}(s)^2} \log(r_1) d^c \psi_1(z_{m-2}) d \varphi_i d \psi_1(z_{m-2}),$$

which cancels with ξ' because $d^c \psi_1(z_n) \wedge d \psi_1(z_n)$ is proportional to $r_n d \varphi_n \wedge d r_n$ and possibly for

$$\frac{1}{h_{\mathcal{E}}(s)^2} \log(r_1) d^c \psi_1(z_{m-2}) d d^c \psi_0(z),$$

But then a $\frac{d \varphi_1}{\log(r_1)}$ must appear somewhere in the expression (35). \square

14. The arithmetic and geometric volume of Shimura varieties of orthogonal type

This section contains the main results of this article. All results are conditional on the existence of a good theory of canonical integral models of toroidal compactifications of Shimura varieties as outlined in the first sections. In the thesis of the author, the existence of such a theory has been proven under a missing technical hypothesis (cf. 6.6).

14.1. Recall the definitions of section 11.5 and 10.15. The relation between $\tilde{\mu}$ and $\tilde{\lambda}$ is given by a global orbit equation:

Theorem 14.2 ([17, Theorem 10.5]). *Assume $m \geq 3$, $m - n \geq 1$. Let D be the discriminant of $L_{\mathbb{Z}}$ and D' be the D -primary part of the discriminant of $M_{\mathbb{Z}}$.*

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\alpha \text{ SO}'(L_{\mathbb{Z}}) \subset \text{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; 0)$$

and

$$\frac{d}{ds} \left(\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \right) \Big|_{s=0} \equiv \sum_{\alpha \text{ SO}'(L_{\mathbb{Z}}) \subset \text{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \frac{d}{ds} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; s) \Big|_{s=0}$$

in $\mathbb{R}_{2DD''}$, where D'' is the product of primes such that $p^2 \nmid D'$.

There is also a relation between different $\tilde{\mu}$'s in the style of Kitaoka's recursion formula:

Theorem 14.3 ([17, Theorem 10.6]). *Assume $m \geq 3$, $m - n \geq 1$. Let D be the discriminant of $L_{\mathbb{Z}}$ and $M_{\mathbb{Z}} = M'_{\mathbb{Z}} \perp M''_{\mathbb{Z}}$. Let D' be the D -primary part of the discriminant of $M'_{\mathbb{Z}}$ (not $M_{\mathbb{Z}}$!). Let $\kappa \in (L'_{\mathbb{Z}}/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^*$ with a corresponding decomposition $\kappa = \kappa' \oplus \kappa''$. We have*

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\substack{\alpha \text{ SO}'(L_{\mathbb{Z}}) \subset \text{I}(M', L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} \tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; 0) \tilde{\mu}(\alpha_{\mathbb{Z}}^{\perp}, M_{\mathbb{Z}}, \kappa''; 0)$$

and

$$\begin{aligned} & \frac{d}{ds} \left(\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \right) \Big|_{s=0} \\ &= \sum_{\substack{\alpha \text{ SO}'(L_{\mathbb{Z}}) \subset \text{I}(M', L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} \frac{d}{ds} \left(\tilde{\lambda}^{-1}(\alpha_{\mathbb{Z}}^{\perp}; s) \tilde{\mu}(\alpha_{\mathbb{Z}}^{\perp}, M'', \kappa''; s) \right) \Big|_{s=0} \end{aligned}$$

in $\mathbb{R}_{2DD'}$. Here $\kappa'' \in (L'_{\mathbb{Z}}/L_{\mathbb{Z}}) \otimes (M''_{\mathbb{Z}})^*$ is considered as an element of $((\alpha_{\mathbb{Z}}^{\perp})^* / \alpha_{\mathbb{Z}}^{\perp}) \otimes (M''_{\mathbb{Z}})^*$ via $\kappa'' \mapsto \kappa'' \cap (\alpha_{\mathbb{A}^{(\infty)}}^{\perp} \otimes (M''_{\mathbb{A}^{(\infty)}})^*)$.

14.4. Let $L_{\mathbb{Z}}$ be a lattice with quadratic form of discriminant $D \neq 0$ and signature $(m-2, 2)$. Let K be the *discriminant kernel* of $L_{\mathbb{Z}}$. It is an admissible compact open subgroup for all $p \nmid D$. Let Δ be a complete and smooth K -admissible rational polyhedral cone decomposition, and let $M(\frac{K}{\Delta} \mathbf{O}(L))$ the *global* canonical model defined over $\mathbb{Z}[1/(2D)]$.

At primes $p \neq 2$, with $p|D$ but $p^2 \nmid D$, we can find a lattice $L'_{\mathbb{Z}}$ containing $L_{\mathbb{Z}}$ as a saturated sublattice of codimension one such that $L'_{\mathbb{Z}(p)}$ is unimodular. If Δ is chosen appropriate, we may define thus a local model over $\mathbb{Z}_{(p)}$ of $M(\frac{K}{\Delta} \mathbf{O}(L))_{\mathbb{Q}}$ at those p by taking normalization and Zariski closure in the model $M(\frac{K'}{\Delta} \mathbf{O}(L'))$ over $\text{spec}(\mathbb{Z}_{(p)})$. We understand by M the so glued global model over $\mathbb{Z}[1/(2D')]$, where D' is the product of primes p such that $p^2|D$. Let $\bar{\mathcal{E}}$ be as before. In the same way, we glue $\Xi^* \bar{\mathcal{E}}$ on M from the pullbacks of $\Xi^* \bar{\mathcal{E}}$ on $M(\frac{K'}{\Delta} \mathbf{O}(L'))$ over $\text{spec}(\mathbb{Z}_{(p)})$. Note that it is not clear from the properties of the theory of canonical models outlined in the first sections, whether this depends on the choice of lattices $L'_{\mathbb{Z}}$. The theorem below, however, is true for any of these choices.

Main Theorem 14.5. *Assume 6.6. We have*

$$\begin{aligned} 1. \quad \text{vol}_E(M) &= 4 \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \\ 2. \quad \widehat{\text{vol}}_{\bar{\mathcal{E}}}(M) &\equiv \frac{d}{ds} 4 \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s) \Big|_{s=0} \quad \text{in } \mathbb{R}_{2D'} \end{aligned}$$

Let $M_{\mathbb{Z}}$ be a lattice of dimension n with positive definite $Q_M \in \text{Sym}^2(M_{\mathbb{Q}}^*)$. Let D'' be the product of primes p such that $M_{\mathbb{Z}_p} \not\subseteq M_{\mathbb{Z}_p}^*$ or $M_{\mathbb{Z}_p}^*/M_{\mathbb{Z}_p}$ is not cyclic. Assume

- $m - n \geq 4$, or
- $m = 4, n = 1$ and $L_{\mathbb{Q}}$ is has Witt rank 1.

Then we have

$$\begin{aligned} 3. \quad \text{vol}_{\mathcal{E}}(\mathbf{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) &= 4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)\tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0), \\ 4. \quad \text{ht}_{\overline{\mathcal{E}}}(\mathbf{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) &\equiv \frac{d}{ds} 4 \left(\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)\tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \right) \Big|_{s=0} \\ &\text{in } \mathbb{R}_{2DD''}. \end{aligned}$$

3. is true without the restriction on m and n . Note that for $n = 1$ and integral Q_M , we have trivially always $D'' = 1$.

Proof. 1. and 3., which are concerned with geometric volumes, are well-known but proofs are included to emphasize the analogies with the arithmetic volume case.

If formula 1. is true for a lattice L then it is true for any lattice L' with $L_{\mathbb{Q}} \cong L'_{\mathbb{Q}}$ respecting the quadratic form up to scalar. This is because in this case the two associated Shimura varieties are related by a correspondence, and for the assertion we only have to show that

$$\frac{[\text{SO}'(L'_{\widehat{\mathbb{Z}}}) : K]}{[\text{SO}'(L_{\widehat{\mathbb{Z}}}) : K]} = \frac{\lambda^{-1}(L'_{\mathbb{Z}}; 0)}{\lambda^{-1}(L_{\mathbb{Z}}; 0)}, \quad (48)$$

where $K \subseteq \text{SO}'(L'_{\widehat{\mathbb{Z}}}) \cap \text{SO}'(L_{\widehat{\mathbb{Z}}})$ is any finite index subgroup. Observe that the morphism Ξ (6.14) is compatible with change of K (see also 10.24). Now $\lambda(L; 0)$ and $\lambda(L'; 0)$ are Euler products, and their factors at almost all p are the same. The finite Euler factors are the volumes of $\text{SO}'(L_{\mathbb{Z}_p})$ and $\text{SO}'(L'_{\mathbb{Z}_p})$, respectively, w.r.t. the canonical measures. This shows (48).

We continue by showing 1. for lattices $L_{\mathbb{Z}}$ with signature $(0, 2)$. Let $L'_{\mathbb{Z}}$ be the lattice $\{A \in M_2(\mathbb{Z}) \mid {}^t A = A\}$ with quadratic form $Q_L(A) = \det(A)$. By the remark before, we may assume w.l.o.g. that the discriminant of $L_{\mathbb{Z}}$ is fundamental, and there is a saturated embedding $L_{\mathbb{Z}} \hookrightarrow L'_{\mathbb{Z}}$. The formula 1. is then basically just the classical Dirichlet class number formula. We prove it here as follows: Let $E'(\tau, s)$ be the Eisenstein series [17, 12.4] und $E(\tau, s) = \frac{Z(2s)}{Z(s)} E'(\tau, s)$. We let $E(L_{\mathbb{Z}}; s)$ be its trace (in the stack sense) over the embedded Shimura variety. By Kronecker's limit formula, we have (using our normalization of the metric h_E , cf. 11.7):

$$2E(L_{\mathbb{Z}}, s+1) = \text{vol}_E(M({}^K \mathbf{O}(L))) + \text{ht}_{\overline{\mathcal{E}}}(M({}^K \mathbf{O}(L)))s + O(s^2).$$

On the other hand, we have [17, Theorem 12.6, 5.]:

$$E(L_{\mathbb{Z}}, s+1) = 2\tilde{\lambda}^{-1}(L'_{\mathbb{Z}}; s)$$

(up to multiplication with a rational function in 2^{-s} with is 1 at $s = 0$). Taking the value at $s = 0$, resp. the derivative at $s = 0$, we get formula 1. — respectively 2. for this special lattice and embedding — in this case.

In the remaining case $m \geq 3$, we give two different proofs of formula 1:

First proof:

Let $M_{\mathbb{Z}} = \langle q \rangle$, $q \in \mathbb{Q}$ and $\kappa \in (L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}^* = L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ is a coset.

We have the following identity:

$$\mu(L_{\mathbb{Z}}, \langle q \rangle, \kappa; 0) = \frac{\text{vol}_E(\mathbf{Z}(L_{\mathbb{Z}}, \langle q \rangle, \kappa; K))}{\text{vol}_E(\mathbf{M}(^K \mathbf{O}(L)))}. \quad (49)$$

This is an application of the Siegel-Weil formula, see [22, 4.17, 4.20, 4.21] and also [18, Theorem 7.6.8]. A certain average over this equation, sufficient for proving 1. by induction follows also from Borchers' theory — see Theorem 13.2, 1. However, this would exclude some special cases. We mention it because it is precisely the geometric analogue of the proof of 2. that we will give below. Comparing 49 to the value at $s = 0$ of the global orbit equation (14.2)

$$\mu(L_{\mathbb{Z}}, \langle q \rangle, \kappa; 0) \lambda(L_{\mathbb{Z}}; 0)^{-1} = \sum_j \lambda((g_j^{-1} x)^{\perp}; 0)^{-1},$$

we establish that assertion 1. is true for lattices of signature $(m-2, 2)$ if and only if it is true for lattices of signature $(m-3, 2)$, whereby it is proven by induction on m because $\sum_i 4\lambda((g_j^{-1} x)^{\perp}; 0)^{-1}$ is $\text{vol}_E(\mathbf{Z}(L_{\mathbb{Z}}, \langle q \rangle, \kappa; K))$ by the induction hypothesis.

Second proof: We assume again $m \geq 3$ and use $\tau(\text{SO}(L_{\mathbb{Q}})) = 2$, where τ means Tamagawa number. We have the following elementary relation:

$$\Lambda \tau = \text{vol}(K) \text{vol}_E(\mathbf{M}(^K \mathbf{O}(L))),$$

where $\Lambda = 2\lambda_{\infty}^{-1}(L; 0)$ is the comparison factor from Lemma 11.17 and $\text{vol}(K)$ is computed with respect to the product of the canonical volumes. Recall that Λ involved the canonical volume form on $\text{SO}(L_{\mathbb{R}})$ and the product over all ν of the canonical measures is a Tamagawa measure ([17, Lemma 10.1]). We have $\text{vol}(K) = \prod_p \lambda_p(L; 0)$ by definition of the λ_p . Everything put together yields $\text{vol}_E(\mathbf{M}(^K \mathbf{O}(L))) = 4\lambda^{-1}(L_{\mathbb{Z}}; 0)$ (which equals also $4\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0)$).

3. follows from 1. using the value of the global orbit equation at $s = 0$:

$$\tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) \tilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\alpha \text{ SO}'(L_{\widehat{\mathbb{Z}}}) \subset \mathbf{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa} \tilde{\lambda}^{-1}(\alpha_{\widehat{\mathbb{Z}}}^{\perp}; 0).$$

2., as an identity $\mathbb{R}_{(2D)}$, follows (using 1.) from the following local statement:

Assertion 14.6. *Let $p \neq 2$. Let $L_{\mathbb{Z}_{(p)}}$ be a unimodular (at p) lattice and signature $(m-2, 2)$. Let K be any admissible compact open subgroup and Δ be complete and smooth w.r.t. K , and let $\mathbf{M} = \mathbf{M}_{\Delta}(^K \mathbf{O}(L))$ the Shimura variety, considered as variety over $\text{spec}(\mathbb{Z}_{(p)})$. We have*

$$\widehat{\text{vol}}_{\widehat{\mathcal{E}}, p}(\mathbf{M}) \equiv \text{vol}_E(\mathbf{M}_{\mathbb{C}}) \left. \frac{\frac{d}{ds} \lambda^{-1}(L_{\mathbb{Z}}; s)}{\lambda^{-1}(L_{\mathbb{Z}}; s)} \right|_{s=0}$$

in $\mathbb{R}^{(p)}$ and any \mathbb{Z} -model $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$.

First of all, to prove Assertion 14.6 we may w.l.o.g. multiply Q_L by a scalar $\in \mathbb{Z}_{(p)}^*$ because the Shimura datum and \mathcal{E} as a SO-equivariant bundle are not affected by this and the Hermitian metric changes by a factor in $\mathbb{Z}_{(p)}^*$ which does not change the arithmetic volume considered in $\mathbb{R}^{(p)}$. Assertion 14.6 also may be shown for any specific p -admissible K and will then be true for any such choice (see 10.24).

The strategy of induction, similar to the first proof of 1., is to walk through the set of (unimodular at p) lattices in a special way, starting from lattices $L_{\mathbb{Z}_{(p)}}$ with known

$$\widehat{\text{vol}}_{\mathcal{E},p}(\text{M}(\Delta^K \mathbf{O}(L_{\mathbb{Z}_{(p)}})))$$

(for admissible K) and then to construct special Borchers products by the Theorems 12.14–12.17 in such a way that all quantities in Theorem 13.2, 2. — except one — are already known. This path through the set of lattices will be mostly according to the dimension and Witt rank of the lattice. It is illustrated by Figure 1, wherein an arrow indicates logical dependence, i.e. the reverse walking direction.

STEP 1: We start with the case of a lattice $L_{\mathbb{Z}_{(p)}}$ of signature $(1, 2)$ and Witt rank 1. These are up to multiplication of Q_L by scalars in $\mathbb{Z}_{(p)}^*$ of the form $Q_L(z) = z_1 z_2 - z_3^2$ which is equivalently the space $\{X \in M_2(\mathbb{Z}_{(p)}) \mid {}^t X = X\}$ with $Q_L(X) = \det(X)$. The arithmetic volume $\widehat{\text{vol}}_{\mathcal{E},p}(\text{M}(\Delta^K \mathbf{O}(L_{\mathbb{Z}_{(p)}})))$ has been calculated in Kühn's thesis [29]. Since multiplying the norm (square root of the metric) by a scalar ρ changes the arithmetic volume of an arithmetic variety X of *arithmetic* dimension $\dim(X)$ by

$$- \dim(X) \log(\rho) \text{ vol},$$

comparing to [29, Theorem 6.1], we see that

$$\widehat{\text{vol}}_{\mathcal{E}}(\text{Sh}^{(K(1))} \mathbf{O}(L)) = \text{vol}_{\mathcal{E}}(\text{Sh}^{(K(1))} \mathbf{O}(L)) \left(-2 \frac{\zeta'(-1)}{\zeta(-1)} - 1 + \log(2) + \log(2\pi) + \gamma \right).$$

In 1. we obtained that

$$\text{vol}_{\mathcal{E}}(\text{Sh}^{(K(1))} \mathbf{O}(L)) = 4\lambda^{-1}(L_{\mathbb{Z}}; 0) = -2\zeta(-1) = \frac{1}{6},$$

which is of course well-known, taking into account that \mathcal{E} is $\mathcal{O}(-2)$ (identifying $\text{M}^{\vee}(\mathbf{O}(L))$ with \mathbb{P}^1), hence $\Xi^*(\mathcal{E})$ is the line-bundle of classical modular forms of weight 2, and $\text{SO}(L)$ is PGL_2 . On the other hand, we have ([17, Example 11.11]):

$$4\tilde{\lambda}^{-1}(L, s) = -2\zeta(-1) - 2\zeta(-1) \left(-2 \frac{\zeta'(-1)}{\zeta(-1)} - 1 + \frac{1}{2} \log(2) + \log(2\pi) + \gamma \right) s + O(s^2).$$

Hence formula 2. is true in \mathbb{R}_2 .

From now on we proceed by induction along the arrows of the diagram according to the following scheme: Choose a saturated $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ such that its discriminant kernel $K \subset \text{SO}(L_{\mathbb{A}(\infty)})$ is neat (it's automatically p -admissible because

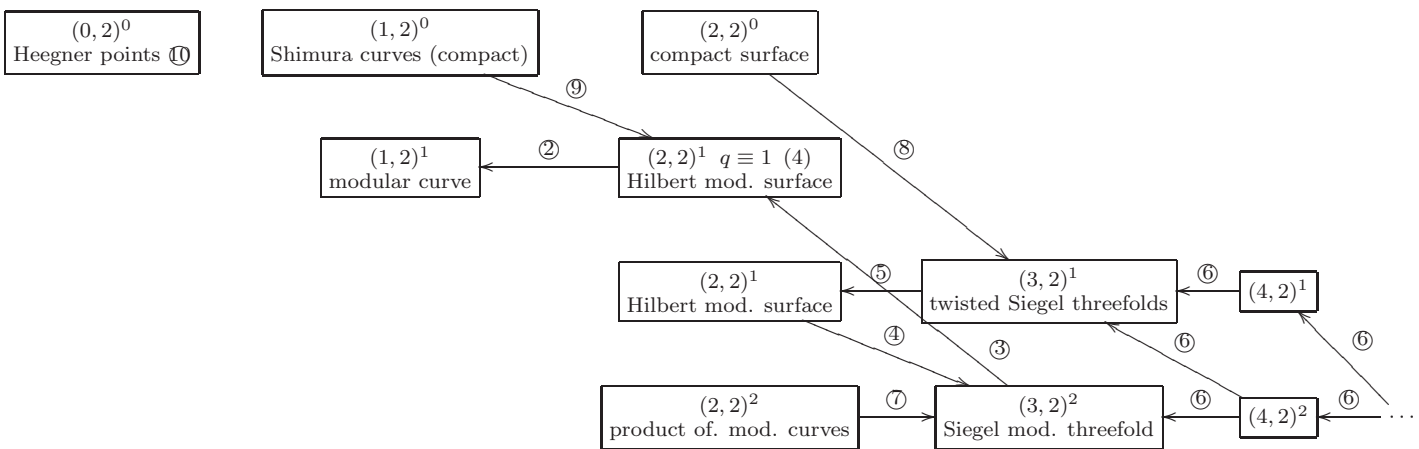


Figure 1: Strategy of induction on the type of lattice

$L_{\mathbb{Z}(p)}$ was assumed to be unimodular). Assume that there is a modular form F of weight $2 - \frac{m}{2}$ associated with the Weil representation of $L_{\mathbb{Z}}$ with q -expansion

$$F(\tau) = \sum_{k \in \mathbb{Q}, k \gg -\infty} a_k q^k$$

with the following properties

- $Z(L, < -k >, a_k; K)$ for all $k < 0$ is a union of (images of) canonical models of sub-Shimura varieties (good reduction),
- $\widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{M}(\frac{K}{\Delta} \mathbf{O}(L')))$ for all these sub-Shimura varieties is already known,
- $a_0(0) \neq 0$.

Note that the first property is equivalent to $p|k \Rightarrow a_k = 0$ for all $k < 0$. Then apply Theorem 13.2, 2. to get

$$\begin{aligned} & \sum_{q < 0} \tilde{\mu}'(L_{\mathbb{Z}}, < -q >, a_q; 0) \text{vol}_E(\text{M}(\frac{K}{\Delta} \mathbf{O}(L))) \\ & + \sum_{q < 0} \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{M}(\frac{K}{\Delta} \mathbf{O}(L))) \\ & = \sum_{q < 0} \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(Z(L_{\mathbb{Z}}, < -q >, a_q; K)). \end{aligned}$$

Inserting the induction hypothesis and 1., we get:

$$\begin{aligned} & \sum_{q < 0} \left(\tilde{\mu}'(L_{\mathbb{Z}}, < -q >, a_q; 0) \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) + \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) \widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{M}(\frac{K}{\Delta} \mathbf{O}(L))) \right. \\ & \quad \left. - \sum_{K \alpha \subseteq \text{I}(< -q >, L_{\mathbb{A}(\infty)}) \cap \text{supp}(a_q)} a_q(\alpha) (\tilde{\lambda}^{-1})'(\alpha_{\mathbb{Z}}^{\perp}; 0) \right) = 0 \end{aligned} \quad (50)$$

The derivative of the global orbit equation 14.2 summed up over all $\kappa \in L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$, $q < 0$ and weighted with $a_q(\kappa)$ yields:

$$\begin{aligned} & \sum_{q < 0} \left(\tilde{\mu}'(L_{\mathbb{Z}}, < -q >, a_q; 0) \tilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0) + \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) (\tilde{\lambda}^{-1})'(L_{\mathbb{Z}}; 0) \right. \\ & \quad \left. - \sum_{K \alpha \subseteq \text{I}(< -q >, L_{\mathbb{A}(\infty)}) \cap \text{supp}(a_q)} a_q(\alpha) (\tilde{\lambda}^{-1})'(\alpha_{\mathbb{Z}}^{\perp}; 0) \right) = 0. \end{aligned} \quad (51)$$

The difference of (50) and (51) gives

$$\begin{aligned} & \sum_{q < 0} \tilde{\mu}(L_{\mathbb{Z}}, < -q >, a_q; 0) \left(\widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{M}(\frac{K}{\Delta} \mathbf{O}(L))) - (\tilde{\lambda}^{-1})'(L_{\mathbb{Z}}; 0) \right) \\ & = a_0(0) \left(\widehat{\text{vol}}_{\overline{\mathcal{E}}, p}(\text{M}(\frac{K}{\Delta} \mathbf{O}(L))) - (\tilde{\lambda}^{-1})'(L_{\mathbb{Z}}; 0) \right) = 0 \end{aligned}$$

using $\sum_{q < 0} \mu(L_{\mathbb{Z}}, < -q >, a_q; 0) = a_0(0)$ (see proof of 13.2, 1.). Since we have $a_0(0) \neq 0$, we get in particular that Assertion 14.6 is true for $L_{\mathbb{Z}}$.

This reduces the proof of Assertion 14.6, and hence 2., to finding appropriate input forms for Borcherds products. This is done for the individual steps in Figure 1 as follows:

STEP 2: Hilbert modular surfaces of prime discriminant $q \equiv 1 \pmod{4}$.

This is the case considered in [8]. We reproduce their argument here as follows: Theorem 12.16, 1. shows that a Borcherds lift of non-zero weight can be found such that all occurring Shimura varieties in $Z(L_{\mathbb{Z}}, < -q >, a_q; K)$ for the occurring a_q are of the form already treated in STEP 1.

STEP 3: Siegel modular threefold. In Theorem 12.17 it is shown that a Borcherds lift of non-zero weight can be found, such that every Shimura variety occurring in $Z(L_{\mathbb{Z}}, < -q >, a_q; K)$ for the non-zero a_q 's is of the form considered in STEP 2 (even copies of the same).

STEP 4: general Hilbert modular surfaces. In Theorem 12.17 it is shown that a Borcherds lift of non-zero weight can be found, such that every Shimura variety occurring in $Z(L_{\mathbb{Z}}, < -q >, a_q; K)$ for the non-zero a_q 's is a $M_{\Delta}^K(\mathbf{O}(L_{\mathbb{Z}_{(p)}}))$ for a given lattice $L_{\mathbb{Z}_{(p)}}$ of signature $(2, 2)$, Witt rank 1. Now $\widehat{\text{vol}}_{\mathcal{E}, p}(M_{\Delta}^K(\mathbf{O}(L)))$ is known by STEP 3, and $\widehat{\text{vol}}_{\mathcal{E}, p}(Z(L_{\mathbb{Z}}, < -q >, a_q; K))$ may be deduced from the comparison of the formula in 13.2 with the derivative of the orbit equation by reverting the argument above.

STEP 5: twisted Siegel threefolds. This is the case L is of signature $(2, 3)$ and Witt rank 1. We have only to avoid $Z(L_{\mathbb{Z}}, < -q >, a_q; K)$'s with occurring Shimura varieties for lattices which are not unimodular at p or compact ones. This is achieved by Theorem 12.15, 1.

STEP 6: dimension 4 and higher. We know all arithmetic volumes for orthogonal Shimura varieties of dimension 3. Hence we may proceed by induction on the dimension and have only to avoid $Z(L_{\mathbb{Z}}, < -q >, a_q; K)$'s containing Shimura varieties for lattices which are not unimodular at p . This is achieved by Theorem 12.14.

We are now left with a couple of cases that have been omitted in the above process. They can be treated analogously to STEP 4 above, by reverting the argument.

STEP 7: product of modular curves. This is the case of signature $(2, 2)$, Witt rank 2. Use Theorem 12.17 again. This can also be treated directly, using STEP 1 (Kühn's thesis), see [10, section 7.8] for a proof.

STEP 8: compact surfaces. This is the case of signature $(2, 2)$, Witt rank 0. Use Theorem 12.15, 2.

STEP 9: Shimura curves. This is the case of signature $(1, 2)$, Witt rank 0. Use Theorem 12.16, 2.

STEP 10: Heegner points. These cases have been treated along the lines of the proof of 1. above.

This completes the proof of Assertion 14.6, and hence of 2., as an identity in \mathbb{R}_{2D} . We use this, and Theorem 13.2 again, to derive the following local assertion, from which 4. follows:

Assertion 14.7. *Let $p \neq 2$. Let $L_{\mathbb{Z}_{(p)}}$ be a unimodular (at p) lattice and of signature $(m-2, 2)$, Let $M_{\mathbb{Z}_{(p)}}$ be a positive definite lattice of dimension n with cyclic $M_{\mathbb{Z}_{(p)}}^*/M_{\mathbb{Z}_{(p)}}$. Assume $m-n \geq 4$, or $m=4, n=1$ and $L_{\mathbb{Q}}$ is isotropic, or $m=5, n=2$ and $L_{\mathbb{Q}}$ has Witt rank 2. Let $L_{\mathbb{Z}}, M_{\mathbb{Z}}$ any saturated lattices in $L_{\mathbb{Z}_{(p)}}$ and $M_{\mathbb{Z}_{(p)}}$, respectively, and let K be any admissible compact open subgroup in the discriminant kernel of $L_{\mathbb{Z}}$. We have:*

$$\begin{aligned} & \tilde{\mu}'(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) \text{vol}_E(\mathbf{M}(\Delta^K \mathbf{O}(L))) \\ & + \deg_E(\mathbf{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) \widehat{\text{vol}}_{\mathcal{F}}(\mathbf{M}(\Delta^K \mathbf{O}(L))) \\ & \equiv \text{ht}_{\mathcal{F}}(\mathbf{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) \end{aligned}$$

in $\mathbb{R}^{(p)}$. Here

$$\deg_E(\mathbf{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) := \frac{\text{vol}_E(\mathbf{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))}{\text{vol}_E(\mathbf{M}(\Delta^K \mathbf{O}(L)))}$$

is the (relative) geometric degree.

We first prove this statement for $n=1$: In this case it follows immediately from Theorem 13.2, 2., using a Borchers product such that all \mathbf{Z} 's in the divisor consist of Shimura varieties corresponding to lattices which are unimodular at p , *except* for $\mathbf{Z}(L_{\mathbb{Z}}, < q >, \kappa; K)$ which shall occur, too, with non-zero multiplicity. Theorem 12.14, 2. enables us to construct such a product. Note that the assumptions imply that $\mathbf{M}(\Delta^K \mathbf{O}(L))$ does have cusps.

Proof for $n \geq 2$: Because $M_{\mathbb{Z}_{(p)}}^*/M_{\mathbb{Z}_{(p)}}$ is cyclic, we may find lattices $M_{\mathbb{Z}}$ and $M'_{\mathbb{Z}}$, unimodular at p , and $< q >_{\mathbb{Z}}$ such that

$$M_{\mathbb{Z}} = < q >_{\mathbb{Z}} \perp M'_{\mathbb{Z}}$$

is a saturated lattice in $M_{\mathbb{Z}_{(p)}}$. Let $\kappa = \kappa_q \oplus \kappa'$ be a corresponding decomposition. Let $\text{SO}'(L_{\mathbb{Z}})\alpha_i \subset \text{I}(L_{\mathbb{Z}}, M'_{\mathbb{Z}}) \cap \kappa'$ be a decomposition into orbits. We may form the cycles $\mathbf{Z}(\alpha_i^{\perp}, < q >, \kappa_q)$ on $\mathbf{M}(\Delta^{K_i} \mathbf{O}(\alpha_i^{\perp}))$, where the K_i are the respective discriminant kernels (all admissible by construction of $M'_{\mathbb{Z}}$). The latter Shimura varieties are all equipped with morphisms into $\mathbf{M}(\Delta^K \mathbf{O}(L_{\mathbb{Z}_{(p)}}))$ (the union of their images is the cycle $\mathbf{Z}(L, M', \kappa', K)$). These Shimura varieties all have the same Shimura datum, and one could see their images as conjugated images of a single Shimura variety with varying K as in the definition of special cycle (9.7). In each of these Shimura varieties, we have cycles $\mathbf{Z}(\alpha_i^{\perp}, < q >, \kappa_q; K_i)$. Identifying them with their image in $\mathbf{M}(\Delta^K \mathbf{O}(L_{\mathbb{Z}_{(p)}}))$, we get

$$\mathbf{Z}(L, M, \kappa; K) = \bigcup_{\text{SO}'(L_{\mathbb{Z}})\alpha \subset \text{I}(L_{\mathbb{Z}}, M'_{\mathbb{Z}}) \cap \kappa'} \mathbf{Z}(\alpha_i^{\perp}, < q >, \kappa_q; K_i).$$

Using the $n=1$ case, we know the height of the right hand side. For this note that the assumptions imply that $\mathbf{M}(\Delta^{K_i} \mathbf{O}(\alpha_i^{\perp}))$ has cusps. Comparing this with global Kitaoka (Theorem 14.3), we get the general result, and its truth does, of course, not depend on the models $L_{\mathbb{Z}}, M_{\mathbb{Z}}$ chosen and not on K (cf. also 10.24).

Finally the previous Assertion 14.6 at $p|D$, where $p \neq 2$ and $p^2 \nmid D$, and therefore 2. as an identity in $\mathbb{R}_{2D'}$ (instead of \mathbb{R}_{2D}) now follows from 14.7 applied to any of the embeddings used to construct the model M (see 14.4). For this, we have to use that the derivative of the global orbit equation 14.2 is also valid for $\tilde{\lambda}$ and $\tilde{\mu}$ in $\mathbb{R}^{(p)}$ for those p . Recall, that this followed from the fact that the local orbit equation for $\tilde{\lambda}_p$, and $\tilde{\mu}_p$ remains true for those p . This is definitely wrong in general if $p^2|D$ (cf. also [17, 11.8]). \square

Remark 14.8. *Main Theorem 14.5, 4. has been proven in full generality, also including information at $2D$ and ∞ in [26–28] for Shimura curves and in [42] for the modular curve ($n = 1$). For the equality at ∞ , with μ_∞ replaced by the full ∞ -factor of the Fourier coefficient of the Eisenstein series, the special cycles have to be complemented by the Kudla-Millson Greens functions (depending on the imaginary part as well), cf. also the introduction to the author’s thesis [18]. There are stronger results also for Hilbert modular surfaces [8, 24, 25], for Siegel threefolds [25], and for the product of modular curves (The Z ’s for $n = 1$ are the Hecke correspondences in this case) in [10, section 7.8].*

A. Lemmas on quadratic forms

Lemma A.1. *Let $p \neq 2$ and $L_{\mathbb{Z}_{(p)}}$ be an unimodular lattice of signature $(2, 2)$ and Witt rank 1, with fundamental discriminant D . Up to multiplication of Q_L by a unit in $\mathbb{Z}_{(p)}^*$, there exists a saturated lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ of the form*

$$L_{\mathbb{Z}} = H \oplus \langle x^2 + xy + \frac{1-D}{4}y^2 \rangle$$

if $D \equiv 1$ modulo 4 or

$$L_{\mathbb{Z}} = H \oplus \langle x^2 - \frac{D}{4}y^2 \rangle$$

if $4|D$.

Proof. We may write $L_{\mathbb{Z}_{(p)}} = H \oplus L'_{\mathbb{Z}_{(p)}}$ and assume that $L'_{\mathbb{Z}_{(p)}}$ represents 1. $L'_{\mathbb{Z}_{(p)}}$ is anisotropic by assumption. Then it is well-known and elementary that $L'_{\mathbb{Z}_{(p)}}$ contains a \mathbb{Z} -lattice of the required form. \square

Lemma A.2. *Let $p \neq 2$ and $L_{\mathbb{Z}_{(p)}}$ be a unimodular anisotropic lattice of signature $(1, 2)$. Up to multiplication of Q_L with a unit in $\mathbb{Z}_{(p)}^*$, there exists a unimodular lattice $L'_{\mathbb{Z}_{(p)}} = L_{\mathbb{Z}_{(p)}} \oplus \langle x \rangle_{\mathbb{Z}_{(p)}}$, $x \in \mathbb{Z}_{>0}$ which is isotropic and of prime fundamental discriminant q , $q \neq p$ and a saturated lattice $L'_{\mathbb{Z}} \subset L'_{\mathbb{Z}_{(p)}}$ of the form*

$$L'_{\mathbb{Z}} = H \oplus \langle x^2 + xy + \frac{1-q}{4}y^2 \rangle.$$

Proof. Write $L_{\mathbb{Z}_{(p)}} = \langle \alpha_1, \dots, \alpha_3 \rangle_{\mathbb{Z}_{(p)}}$, with $\alpha_i \in \mathbb{Z}$ square-free and relatively prime. This is possible without multiplying Q_L by a scalar. Let D be the square-free part of $\prod \alpha_i$. We may find a prime q , different from p , such that $q \equiv 1 \pmod{4}$ and for any $l \neq 2$ with $\nu_l(\prod \alpha_i) = 2$ (hence $l \nmid D$), we have $Dq \equiv -\alpha_j \pmod{l}$, where α_j is the one not divisible by l .

We may also prescribe its residue mod 8, such that $L_{\mathbb{Q}_2} \oplus \langle Dq \rangle$ is isotropic. For if they were anisotropic for q, q' congruent to 1, 5 modulo 8 respectively, we would get $\langle Dq \rangle_{\mathbb{Q}_2} \cong \langle Dq' \rangle_{\mathbb{Q}_2}$ from Witt's Theorem and the uniqueness of the 4-dimensional anisotropic space, which is absurd.

Hence

$$L_{\mathbb{Z}_{(p)}} \oplus \langle Dq \rangle_{\mathbb{Z}_{(p)}}$$

is unimodular, of signature (2, 2) with square-free discriminant q . It is isotropic at all $l \neq 2$, too, because of the congruence condition on Dq . Hence the Witt rank is 1 and we may apply the previous lemma to it (this changes also the form on $L_{\mathbb{Z}_{(p)}}$ by a scalar) to get the result. \square

Lemma A.3. *Let $p \neq 2$ and $L_{\mathbb{Z}_{(p)}}$ be an unimodular anisotropic lattice of signature (1, 2). Up to multiplication of Q_L with a unit in $\mathbb{Z}_{(p)}^*$, there is a saturated lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ of discriminant $D = 2D'$, where D' is square-free and with $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ cyclic.*

It has the property that a primitive $\kappa \in L_{\mathbb{Z}}^/L_{\mathbb{Z}}$ represents an $m \in \mathbb{Q}$ if and only if $Q_L(\kappa) \equiv m \pmod{1}$.*

Proof. First we prove the existence of the lattice. Write $L_{\mathbb{Z}_{(p)}} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathbb{Z}_{(p)}}$, with $\alpha_i \in \mathbb{Z}$ square-free. By multiplying Q_L , if necessary, by a scalar, we may assume that the α_i are pairwise relatively coprime.

The lattice $L_{\mathbb{Z}} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathbb{Z}}$ then already satisfies the assumption locally for all $l \neq 2$. It suffices to construct a lattice $L_{\mathbb{Z}_2}$ satisfying the assumption. There are 2 cases:

1. The space $L_{\mathbb{Q}_2}$ is isotropic, hence there is, up to multiplication of the quadratic form by 2, a lattice of the form

$$L_{\mathbb{Z}_2} = H \perp \langle 1 \rangle,$$

which has discriminant 2.

2. The space $L_{\mathbb{Q}_2}$ is anisotropic, hence, up to multiplication of the lattice by ± 2 , of the form

$$L_{\mathbb{Q}_2} = \langle 1, 3, 2\epsilon \rangle$$

with $\epsilon \in \{1, 3, -3, -1\}$. In it there exists the lattice

$$L_{\mathbb{Z}_2} = \langle x^2 + xy + y^2 \rangle \perp \langle 2\epsilon \rangle$$

of discriminant 4, with $L_{\mathbb{Z}_2}^*/L_{\mathbb{Z}_2}$ cyclic of order 4.

The only-if part of the claimed property is clear. For the if part, we have to show that m is represented by $L_{\mathbb{Z}_l}$ for all l . For $l \nmid D$, $L_{\mathbb{Z}_l}$ represents every $m \in \mathbb{Z}_l$. For $l \mid D, l \neq 2$, κ_l represents m if

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 \left(\left(\frac{\beta}{l} \right)^2 + 2 \frac{\beta x_3}{l} + x_3^2 \right) = \frac{n}{l} = m$$

has a solution with $x_i \in \mathbb{Z}_l$, where $\beta \in \mathbb{Z}_l^*$ is determined by κ and $n \in \mathbb{Z}_l^*$. Here w.l.o.g. $l|\alpha_3$.

Now the condition given implies $\frac{\alpha_3}{l}\beta^2 \equiv n \pmod{l}$. Since $\alpha_1 x_1^2 + \alpha_2 x_2^2$ represents \mathbb{Z}_l^* , κ_l represents m (choose x_3 appropriately in the above equation).

For $l = 2$, case 1, κ_2 represents m if

$$x_1 x_2 + ((\frac{1}{2})^2 + x_3 + x_3^2) = \frac{n}{4} = m$$

has a solution with $x_i \in \mathbb{Z}_l$. This follows from the condition which boils down to $n \equiv 1 \pmod{4}$ in this case.

For $l = 2$, case 2, κ_2 represents m if

$$x_1^2 + x_1 x_2 + x_2^2 + 2\epsilon((\frac{\beta}{4})^2 + 2\frac{\beta}{4}x_3 + x_3^2) = \frac{n}{8} = m$$

has a solution with $x_i \in \mathbb{Z}_l$. Here $\beta = \pm 1$ distinguishes the two κ_2 . The condition says $\beta^2 \epsilon \equiv n \pmod{8}$, hence solubility because $x_1^2 + x_1 x_2 + x_2^2$ represents \mathbb{Z}_2^* . \square

The strong form of the lemma is wrong for lattices of dimension 2. However, we have the following weaker form:

Lemma A.4. *Let $p \neq 2$ and $L_{\mathbb{Z}(p)}$ be a unimodular anisotropic lattice of signature $(1, 1)$ and of discriminant $-q$, where $q \equiv 1 \pmod{4}$ is prime. Up to multiplication of Q_L by a unit $\in \mathbb{Z}_{(p)}^*$, there is a saturated lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}(p)}$ of discriminant q of the form*

$$L_{\mathbb{Z}} = \langle x^2 + xy + \frac{1-q}{4}y^2 \rangle.$$

It has the property that a primitive $\kappa \in L_{\mathbb{Z}}^/L_{\mathbb{Z}}$ represents a $m = \frac{l'}{q} \in \mathbb{Q}$, where $l' \neq 2$ is prime, if and only if $Q_L(\kappa) \equiv m \pmod{1}$.*

Proof. The existence of the lattice is well-known (compare also A.1). For the property, we have to show, that $L_{\mathbb{Z}_l}$ represents $\frac{l'}{q}$ under the condition.

We can write $L_{\mathbb{Z}_l} = \langle 1, -q \rangle$, hence we see (like in the proof of the last lemma) that the condition implies $-x^2 \equiv l' \pmod{q}$ has a solution, i.e. $-l'$ is a quadratic residue mod q . Since $q \equiv 1 \pmod{4}$ this is equivalent to l' being a quadratic residue mod q .

For $l \notin \{2, q\}$,

$$x_1^2 - qx_2^2 = \frac{l'}{q}$$

clearly has a solution for $x_1, x_2 \in \mathbb{Z}_l$ if $l \neq l'$, and for $l = l'$ it does, if q is a square mod l' , and the latter is true by the law of quadratic reciprocity. For $l = q$, κ_q represents m , if

$$x_1^2 - q((\frac{\beta}{q})^2 + 2\frac{\beta}{q}x_2 + x_2^2) = \frac{l'}{q}$$

has a solution with $x_i \in \mathbb{Z}_q$. Now $-\beta^2 \equiv l' \pmod{q}$, so this is clearly possible.

For $l = 2$, we have $L_{\mathbb{Z}_2} = \langle x^2 + xy + z^2 \rangle$, which represents \mathbb{Z}_2^* . \square

Lemma A.5. *Let $p \neq 2$ and $L_{\mathbb{Z}_{(p)}}$ be a unimodular anisotropic lattice of signature $(2, 2)$. Up to multiplication of Q_L with a unit in $\mathbb{Z}_{(p)}^*$, there exists a unimodular lattice $L'_{\mathbb{Z}_{(p)}} = L_{\mathbb{Z}_{(p)}} \oplus \langle x \rangle$ of signature $(3, 2)$ and a saturated \mathbb{Z} -lattice $L'_\mathbb{Z} \subset L'_{\mathbb{Z}_{(p)}}$ of discriminant $D = 2D'$, where D' is square-free of the form*

$$L'_\mathbb{Z} = H \oplus L''_\mathbb{Z}.$$

such that $(L'_\mathbb{Z})^/L'_\mathbb{Z}$ is cyclic.*

Proof. Take $L'_{\mathbb{Z}_{(p)}} = L_{\mathbb{Z}_{(p)}} \oplus \langle x \rangle$ for an arbitrary positive $x \in \mathbb{Z}_{(p)}^*$. It is automatically isotropic, hence of the form

$$L'_{\mathbb{Z}_{(p)}} = H \oplus L''_{\mathbb{Z}_{(p)}}.$$

Applying Lemma A.3 to $L''_{\mathbb{Z}_{(p)}}$, we get the result (we multiply $Q_{L'}$ by the same scalar; this multiplies also Q_L on $L_{\mathbb{Z}_{(p)}}$ by this scalar — note, however, that multiplying the quadratic form on H by a scalar does not affect its isomorphism class). \square

B. Lacunarity of modular forms

Lemma B.1. *Let $N > 0$ be an integer and p a prime with $p \nmid N$. Let f be a holomorphic modular form of (half-)integral weight $k \neq 0$ for the group $\Gamma(N)$. If f has a Fourier expansion of the form*

$$f = \sum_{n \in \mathbb{Q}_{\geq 0}} a_n q^n,$$

where a_n is zero, unless n is an integral multiple of $\frac{p}{N}$, then $f = 0$.

Proof. The assumption implies that f is periodic with period $\frac{N}{p}$. It is hence a modular form for the group Γ , generated by $\Gamma(N)$ and the matrix $\begin{pmatrix} 1 & \frac{N}{p} \\ 0 & 1 \end{pmatrix}$. This group contains the product

$$C = \begin{pmatrix} 1 & \frac{N}{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{N^2}{p} & \frac{N}{p} \\ N & 1 \end{pmatrix}.$$

The trace of C is equal to $2 + \frac{N^2}{p}$. Since $p \nmid N$, its p -adic valuation is > 1 . Hence at least one of the eigenvalues of C has p -adic valuation > 1 . (We choose some fixed extension of the p -adic valuation to $\overline{\mathbb{Q}}$). It follows that $\|C^i\| \rightarrow \infty$ for any chosen p -adic matrix norm $\|\cdot\|$. From this, it follows that $[\Gamma : \Gamma(N)] = \infty$. For assume that there are finitely many representatives α_i . Let ν be the maximum of their p -adic matrix norms. Every element $\gamma \in \Gamma$ is of the form

$$\alpha_i \gamma'$$

for $\gamma' \in \Gamma(N)$. The matrix norm of γ is hence $\leq \nu$. A contradiction. Hence Γ cannot be a discrete subgroup, and since $k \neq 0$, we have $f = 0$. \square

Proposition B.2. *Let $q \equiv 1 \pmod{4}$ be a prime and S a finite set of primes. Let $\chi_q(x) = \left(\frac{x}{q}\right)$ and $k \geq 2$ be an integer. If $f \in \text{HolModForm}(\Gamma_0(q), \chi_q, k)$ has a Fourier expansion of the form*

$$f = \sum_{n \in \mathbb{Z}_{\geq 0}} a_n q^n,$$

with algebraic a_n , where (i) $a_n = 0$, whenever $\chi_q(n) = -1$, and (ii) $a_p = 0$, whenever $p \notin S$. Then $f = 0$.

Proof. This follows from an idea of [35], see also [8, Lemma 4.14]: Since $q \equiv 1 \pmod{4}$, there are no forms of type ‘CM’ satisfying condition (i). Since the a_n are algebraic, we can write f as a linear combination

$$f = c_0 E_0 + c_\infty E_\infty + \sum_{i=1}^n c_i f_i,$$

where the c_i are algebraic, the f_i are cuspidal Hecke eigenforms and E_0, E_∞ are the Eisenstein series.

In [35, Lemma 1] it is shown that for all sufficiently large primes l the image of the mod l representation $\rho := \rho_{1,l} \times \cdots \times \rho_{n,l}$ contains a subgroup G conjugated to

$$\text{SL}_2(\mathbb{F}_1) \times \cdots \times \text{SL}_2(\mathbb{F}_n)$$

and $\text{im}(\rho)/G$ is Abelian, where the \mathbb{F}_i ’s are defined in [loc. cit.]. Choose l such that all $|c_i|_l = 1$, whenever $c_i \neq 0$.

The mod l representations $\rho_{0,l}$, resp. $\rho_{\infty,l}$ of E_0 , resp. E_∞ [37, p. 28] are abelian and $\rho' := (\rho_{0,l} \times \rho_{\infty,l}) \times (\rho_{1,l} \times \cdots \times \rho_{n,l})$ contains a subgroup of the form

$$H \times G$$

because $\text{SL}_2(\mathbb{F}_i)$ has no non-trivial Abelian quotient if $l > 4$. Here $H \subset \text{im}(\rho_{0,l} \times \rho_{\infty,l})$ is a subgroup that, at least, contains a pair of matrices with different traces. Choose a pair $(M_1, M_2) \in H$ of matrices with different or equal traces, according to whether $c_0 \equiv -c_\infty \pmod{l}$ or not.

By Čebotarev, there is a positive density of primes p , such that the image of Frob_p is conjugated to

$$A = M_1 \times M_2 \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence $|a_p(c_0 E_0 + c_\infty E_\infty)|_l = 1$ and $|a_p(f_i)|_l < 1$. We get $a_p(f) \neq 0$ for infinitely many primes, a contradiction. If $c_0 = c_\infty = 0$, and $c_1 \neq 0$ (say) choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and argue as before. \square

Lemma B.3. *Let $\Gamma \subset \text{Mp}_2(\mathbb{R})$ be an arithmetic subgroup and V_ρ, ρ an irreducible representation of Γ . Let $f \in \text{HolModForm}(\Gamma, \rho, k)$ and $\beta : V_\rho \rightarrow \mathbb{C}$ be a non-zero linear form. If $\beta \circ f = 0$ then $f = 0$.*

Proof. This follows immediately from the irreducibility: Choose a basis $\{e_i\}$ of V_ρ such that $\beta = e_0^*$. Since V_ρ is irreducible, for any i there is an operator of the form

$$O_i = \sum_j \alpha_j \rho(\gamma_j),$$

which interchanges e_i and e_0 . Consider the form $O_i \circ f$. $e_0^* \circ O_i \circ f$ is equal to $e_i^* \circ f$ on the one hand. On the other hand it is equal to $\sum_i \alpha_i (e_0^* \circ f)|_k \gamma_i$, which is zero by assumption. Hence $f = 0$. \square

C. Semi-linear representations

Let K be a pro-finite topological group and $\tau : K \rightarrow \text{Gal}(F : \mathbb{Q})$ be a continuous homomorphism, where F is a (possibly infinite) algebraic extension of \mathbb{Q} .

Definition C.1. *By a τ -semi-linear representation of K on a finite-dimensional F -vector-space V , we understand a continuous homomorphism*

$$\omega : K \rightarrow \text{Aut}_{\mathbb{Q}}(V)$$

(where V is equipped with the discrete topology) satisfying

$$\omega(g)(\alpha v) = \tau_g(\alpha) \omega(g)v.$$

Example C.2. *For example, a τ -semi-linear representation may be given by a \mathbb{Q} -vector space V and a continuous homomorphism ω , fitting into the following commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{\omega} & \text{GL}_F(V_F) \rtimes \text{Gal}(F : \mathbb{Q}) \\ & \searrow \tau & \downarrow \\ & & \text{Gal}(F : \mathbb{Q}) \end{array} \quad (\text{C.1})$$

Obvious any such homomorphism determines and is determined by its restriction ω' to the first factor, which is an element in $H^1(K, \text{GL}_F(V_F))$, where K acts via τ . Here $\text{GL}_F(V_F)$ is equipped with the discrete topology.

C.3. τ -semi-linear representations of K form a category, morphisms being linear morphisms of F -vector-spaces respecting the representations.

There is a notion of induction for these representations, and we have for $K_1 \subseteq K_2$ (open subgroup) and V_i a τ -semi-linear representation of K_i :

$$\text{Hom}_{K_1}(V_2, V_1) \cong \text{Hom}_{K_2}(V_2, \text{ind}_{K_1}^{K_2}(V_1))$$

naturally, and since $[K_1 : K_2] < \infty$, we also have

$$\mathrm{Hom}_{K_1}(V_1, V_2) \cong \mathrm{Hom}_{K_2}(\mathrm{ind}_{K_1}^{K_2}(V_1), V_2),$$

naturally. The induction may be defined as

$$\mathrm{ind}_{K_1}^{K_2}(V) = \{f : K_2 \rightarrow V \mid f(hg) = \omega(h)f(g) \ \forall h \in K_1, g \in K_2\}$$

with the F -linear structure given by

$$(\alpha f)(g) := \tau_g(\alpha)f(g).$$

We have also

$$\begin{aligned} & \mathrm{Hom}_{K_2}(\mathrm{ind}_{K_1}^{K_2}(V), \mathrm{ind}_{K_1}^{K_2}(W)) = \\ & \{\varphi : K_2 \rightarrow \mathrm{Hom}_{\mathbb{Q}}(V, W) \mid \varphi(h_1gh_2) = \omega(h_1) \circ \varphi(g) \circ \omega(h_2) \ \forall h_1, h_2 \in K_1, g \in K_2, \\ & \quad \varphi(g)\alpha v = \tau_g(\alpha)\varphi(g)v \ \forall g \in K_2, \alpha \in F\}. \end{aligned}$$

Here a function φ acts by convolution

$$(\varphi * f)(g) := \frac{1}{[K_2 : K_1]} \sum_{x \in K_2 \setminus K_1} \varphi(gx^{-1})f(x).$$

Example C.4. Any \mathbb{Q} -representation ω', V of K gives rise to a τ -semi-linear representation $\omega(k) = \omega'(k) \circ \tau_k$ on V_F .

For a τ -semi-linear representation V, ω the dual is defined by V^* (usual dual) acted on by $(\omega^*(g)v^*)(v) := \tau_g(v^*(\omega(g^{-1})v))$. It is a τ -semi-linear representation, too.

Example C.5. Let p be an odd prime, χ be the non-trivial character of order 2 of $(\mathbb{Z}/p\mathbb{Z})^*$ and $\tau : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ be the natural isomorphism. It gives rise to a semi-linear representation

$$\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto \chi(a)\tau_{ad}.$$

A function $\varphi \in \mathrm{Hom}(\mathrm{ind}_{K_1}^{K_2}\chi, \mathrm{ind}_{K_1}^{K_2}\chi)$ is determined by its values at

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The semi-linearity forces

$$\tau_\alpha(\varphi(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})) = \varphi(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \quad \text{and} \quad \tau_\alpha(\varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})) = \chi(\alpha)\varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}),$$

therefore

$$\varphi(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \in \mathbb{Q} \quad \text{and} \quad \varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \in \mathbb{Q} \cdot \sqrt{p^*},$$

where $p^* = p$ if $p \equiv 1 \pmod{4}$ and $p^* = -p$ otherwise. A calculation shows that $\mathrm{End}_{K_2}(\mathrm{ind}_{K_1}^{K_2}\chi) = \mathbb{Q}^2$ as algebras, hence the irreducible constituents of $\mathrm{ind}_{K_1}^{K_2}\chi$ as $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ -representation are also irreducible for this semi-linear representation of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

References

- [1] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. *Smooth compactification of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.
- [2] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966. ISSN 0003-486X.
- [3] R. E. Borcherds. Automorphic forms with singularities on Grassmannians. *Invent. Math.*, 132(3):491–562, 1998. ISSN 0020-9910.
- [4] R. E. Borcherds. The Gross-Kohnen-Zagier theorem in higher dimensions. *Duke Math. J.*, 97(2):219–233, 1999. ISSN 0012-7094.
- [5] N. Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. ISBN 3-540-19371-5. Translated from the French, Reprint of the 1972 edition.
- [6] J. H. Bruinier. Two applications of the curve lemma for orthogonal groups. *Math. Nachr.*, 274/275:19–31, 2004. ISSN 0025-584X.
- [7] J. H. Bruinier and U. Kühn. Integrals of automorphic Green’s functions associated to Heegner divisors. *Int. Math. Res. Not.*, 31:1687–1729, 2003. ISSN 1073-7928.
- [8] J. H. Bruinier, J. I. Burgos Gil, and U. Kühn. Borcherds products and arithmetic intersection theory on Hilbert modular surfaces. *Duke Math. J.*, 139(1):1–88, 2007. ISSN 0012-7094.
- [9] J. I. Burgos Gil, J. Kramer, and U. Kühn. Arithmetic characteristic classes of automorphic vector bundles. *Doc. Math.*, 10:619–716 (electronic), 2005. ISSN 1431-0635.
- [10] J. I. Burgos Gil, J. Kramer, and U. Kühn. Cohomological arithmetic Chow rings. *J. Inst. Math. Jussieu*, 6(1):1–172, 2007. ISSN 1474-7480.
- [11] P. Deligne. Espaces hermitiens symétriques. transcription available at <http://www.jmilne.org/>, 1973.
- [12] G. Faltings and C.-L. Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990. ISBN 3-540-52015-5. With an appendix by David Mumford.
- [13] H. Gillet. Arithmetic intersection theory on Deligne-Mumford stacks. In *Motives and algebraic cycles*, volume 56 of *Fields Inst. Commun.*, pages 93–109. Amer. Math. Soc., Providence, RI, 2009.

- [14] B. H. Gross and D. B. Zagier. Heegner points and derivatives of L -series. *Invent. Math.*, 84(2):225–320, 1986. ISSN 0020-9910.
- [15] A. Grothendieck et al. Séminaire de Géométrie algébrique du Bois-Marie III, Schémas en Groupes, 1962–1964.
- [16] M. Harris and S. Zucker. Boundary cohomology of Shimura varieties. I. Coherent cohomology on toroidal compactifications. *Ann. Sci. École Norm. Sup. (4)*, 27(3):249–344, 1994. ISSN 0012-9593.
- [17] F. Hörmann. On recursive properties of certain p -adic Whittaker functions. submitted for publication, arXiv:1010.1009v1, 2010.
- [18] F. Hörmann. *The arithmetic volume of Shimura varieties of orthogonal type*. PhD thesis, Humboldt-Universität zu Berlin, 2010.
- [19] M. Kisin. Integral canonical models of Shimura varieties. *J. Théor. Nombres Bordeaux*, 21(2):301–312, 2009.
- [20] M. Kisin. Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 23(4):967–1012, 2010.
- [21] K. Köhler. A Hirzebruch proportionality principle in Arakelov geometry. In *Number fields and function fields—two parallel worlds*, volume 239 of *Progr. Math.*, pages 237–268. Birkhäuser Boston, Boston, MA, 2005.
- [22] S. S. Kudla. Integrals of Borcherds forms. *Compositio Math.*, 137(3):293–349, 2003. ISSN 0010-437X.
- [23] S. S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin L -series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004.
- [24] S. S. Kudla and M. Rapoport. Arithmetic Hirzebruch-Zagier cycles. *J. Reine Angew. Math.*, 515:155–244, 1999. ISSN 0075-4102.
- [25] S. S. Kudla and M. Rapoport. Cycles on Siegel threefolds and derivatives of Eisenstein series. *Ann. Sci. École Norm. Sup. (4)*, 33(5):695–756, 2000. ISSN 0012-9593.
- [26] S. S. Kudla, M. Rapoport, and T. Yang. On the derivative of an Eisenstein series of weight one. *Internat. Math. Res. Notices*, 7:347–385, 1999. ISSN 1073-7928.
- [27] S. S. Kudla, M. Rapoport, and T. Yang. Derivatives of Eisenstein series and Faltings heights. *Compos. Math.*, 140(4):887–951, 2004. ISSN 0010-437X.
- [28] S. S. Kudla, M. Rapoport, and T. Yang. *Modular forms and special cycles on Shimura curves*, volume 161 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2006. ISBN 978-0-691-12551-0; 0-691-12551-1.

- [29] U. Kühn. Generalized arithmetic intersection numbers. *J. Reine Angew. Math.*, 534:209–236, 2001. ISSN 0075-4102.
- [30] K.-W. Lan. *Arithmetic compactifications of PEL-type Shimura varieties*. PhD thesis, Harvard University, 2008.
- [31] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005.
- [32] J. S. Milne and K.-y. Shih. Shimura varieties: conjugates and the action of complex multiplication. available on <http://www.jmilne.org/>, 1979.
- [33] B. Moonen. Models of Shimura varieties in mixed characteristics. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 267–350. Cambridge Univ. Press, Cambridge, 1998.
- [34] D. Mumford. Hirzebruch’s proportionality theorem in the noncompact case. *Invent. Math.*, 42:239–272, 1977. ISSN 0020-9910.
- [35] K. Ono and C. Skinner. Fourier coefficients of half-integral weight modular forms modulo l . *Ann. of Math. (2)*, 147(2):453–470, 1998. ISSN 0003-486X.
- [36] R. Pink. *Arithmetical compactification of mixed Shimura varieties*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 209. Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.
- [37] K. A. Ribet. Galois representations attached to eigenforms with Nebentypus. In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 17–51. Lecture Notes in Math., Vol. 601. Springer, Berlin, 1977.
- [38] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. ISBN 0-521-41669-8. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.
- [39] A. Vasiu. Integral canonical models of Shimura varieties of preabelian type. *Asian J. Math.*, 3(2):401–518, 1999. ISSN 1093-6106.
- [40] A. Weil. Sur certains groupes d’opérateurs unitaires. *Acta. Math.*, 111: 143–211, 1964.
- [41] A. Weil. Sur la formule de Siegel dans la théorie des groupes classiques. *Acta. Math.*, 113:1–87, 1965.
- [42] T. Yang. Faltings heights and the derivative of Zagier’s Eisenstein series. In *Heegner Points and Rankin L-Series*, volume 49. MSRI Publications, 2004.